General
Zoulou

ISIS

International Review devoted to the History
:: of Science and Civilization ::
Quarterly Organ of the History of Science Society and of the International Committee of the History of Science.

SUMMARY
of No 58 (Vol. XX, 1)
320 p., 4 pl.

I. — *Main articles* (8)

G. SARTON *(Cambridge, Mass.)*.—The teaching and study of the history of science at the University of California 6
P. LUCKEY *(Berlin-Dahlem)*.—Was ist ägyptische Geometrie? . 15
R. STEELE *(London)*.—ROGER BACON as Professor. A student's notes 53
W. A. OLDFATHER, C. A. ELLIS, D. M. BROWN *(Urbana, Ill.)*.—LEONHARD EULER's Elastic curves, translated and annotated (4 pl.) 72
D. J. STRUIK *(Cambridge, Mass.)*.—Outline of a history of differential geometry. II 161
G. WINDRED *(Bedford)*.—The history of mathematical time. II 192
D. KOTSOVSKY *(Chisinau)*.—Das Alter in der Geschichte der Wissenschaft 220
A. C. BURR *(Cambridge, Mass.)*.—Notes on the history of the concept of thermal conductivity 246

II. — *Notes and correspondence* (6)

Queries: Editorial Preface (G. SARTON).—Arabic "commercial" arithmetic (G. S.).—Hippocratic oath in Arabic (G. S.).—Orientation of the miḥrāb in mosques (G. S.).—Le Prospectus de l'Encyclopédie (1750) (G. S.) 260
Miscellanea: Cours à l'Institut d'histoire des sciences, Paris, 1932-1933. — Centenaire de la mort de NICÉPHORE NIEPCE . . . 265

III. — *Reviews* (19)

IInd Century PTOLEMY. Geographiae Codex Urbinas Graecus 82. Ed. J. FISCHER. 4 vols., Leiden, BRILL, 1932 (G. S.) 267

PTOLEMY. Geography. Transl. by E. L. STEVENSON. New York, Public Library, 1932 (G. S.) 270

GANDZ, S. The Mishnat ha-Middot. Hebrew and Arabic texts, with introduction, translation and notes. Berlin, SPRINGER, 1932 (W. THOMSON) 274

Xth Century. ERLANGER, R. d'. La musique arabe. T. I. AL-FĀRĀBĪ. Paris, GEUTHNER, 1930 (G. S.) 280

IBN AL-NADĪM. Al-fihrist. Cairo, S. M. 'ALI, 1930 (G. S.) 283

XVIIth Century. TIERIE, G. CORNELIS DREBBEL. Amsterdam, PARIS, 1932 (T. L. D.) 285

STURT, M. FRANCIS BACON, a biography. New York, MORROW, 1932 (C. A. K.) . . 286

XVIIIth Century. HELMHOLTZ, H. v., DOHRN, M., SCHIFF, J. GOETHES naturwissenschaftliches Denken und Wirken. Berlin, SPRINGER, 1932 (E. WOLF) 287

XIXth Century. OSBORN, H. F. COPE : master naturalist. Princeton University Press, 1931 (C. A. KOFOID) 291

STILLMAN, C. G. SAMUEL BUTLER, a mid-Victorian modern. New York, The Viking Press, 1932 (C. A. K.) 294

PETERSON, H. HUXLEY, prophet of science. London, LONGMANS, 1932 (C. A. K.) . . 297

AYRES, C. HUXLEY. New York, NORTON, 1932 (C. A. K.) 298

XXth Century. PAUL, J. H. The last cruise of the Carnegie. Baltimore, WILLIAMS and WILKINS, 1932 (G. S.) 300

Middle Ages. RUSKA, J. Turba philosophorum : ein Beitrag zur Geschichte der Alchemie. Berlin, SPRINGER, 1931 (E. J. HOLMYARD) . . . 302

Japan. OGURA, K. History of mathematical education. Tokyo, SHOTEN, 1932 (S. IKEHARA) 305

Science, Philosophy. MEAD, G. H. The philosophy of the present. Chicago, Open Court, 1932 (H. T. DAVIS) 307

Astronomy. GUNTHER, R. T. The astrolabes of the world. Vol. I. Oxford University Press, 1932 (A. POGO) 310

SITTER, W. DE. Kosmos. Harvard University Press, 1932 (N. T. BOBROVNIKOFF) . . . 316

Biology. WODLINGER, M. H. Bibliographical survey of vitamins, 1650-1930. Chicago, The John Crerar Library, 1932 (C. A. K.). . . . 319

The Teaching and Study of the History of Science at the University of California

Having just returned from California, where I spent five delightful weeks (from Jan. 4th to February 7th, 1933), it occurred to me that a report of my experiences and impressions might be of interest to our readers.

It is simpler to state my conclusion at once : No university, certainly no American university, has done as much to promote the study of the History of Science as the University of California. I think the following facts will make its leadership sufficiently clear.

Let us first consider the regular courses on the subject. The following list based on information kindly given by Professor LARKEY is so impressive that it requires but little commentary.

Mathematics.—A special chair devoted to the history of mathematics was created in 1918 in favor of the late FLORIAN CAJORI (1), who held it until 1929 when he became emeritus. This was one of very few chairs of its kind in the world. One could not be too grateful to the University of California for that creation which enabled Professor CAJORI to accomplish some of his best work. By the way, this special creation suggests one of the simplest ways of solving some of our problems. It may be too onerous or too risky to create permanent chairs, but universities would honor themselves by establishing temporary chairs making it possible for historians of science of proved merit and fecundity to fulfill more completely their purpose and to deliver more effectively their message to the new generation.

---

(1) R. C. ARCHIBALD : FLORIAN CAJORI, 1859-1930, (*Isis* 17, 384-407, 1932) with portrait and bibliography.

A course on the history of mathematics (two hours a week yearly) is now given by Dr. D. C. DUNCAN.

Astronomy.—A course on the history of astronomy is given by Professor W. F. MEYER.

Physics.—Associate Professor VICTOR F. LENZEN offers a course on the history of physics. He discusses the fundamental concepts and principles of mechanics, heat, electricity and magnetism, radiation and atomic structure.

Chemistry.—The history of chemistry is taught by Professor W. C. BLASDALE.

Geology.—The history of geology, by Associate Professor N. L. TALIAFERRO.

Geography.—A seminar in the history of geography is conducted by Assistant Professor J. B. LEIGHLY. The topic for 1933 is the emergence of scientific geography in the nineteenth century.

Botany.—The history of botany has been taught by Professor W. A. SETCHELL for so many years that he must be considered one of our pioneers. He became professor of botany at the University of California in 1895 succeeding another pioneer of botanical history, that very strange character, EDWARD LEE GREENE (1843-1915) (2). Professor SETCHELL's interesting syllabi have been registered in our critical bibliography.

Zoology.—A kind of seminar course is conducted by Professor C. A. KOFOID. It is given conjointly by the departments of anatomy, bacteriology, history of medicine, pharmacology, physiology, and zoology.

Entomology.—The history of entomology is taught by Professor E. O. ESSIG, whose treatise on the subject : *A history of entomology* (New York, 1931) was reviewed in *Isis*, vol. 17, 447-50.

Psychology.—A course on the history of psychology is offered by Professor G. M. STRATTON. Phases of ancient, medieval, and modern psychology with particular attention to the nineteenth and twentieth centuries.

Medicine.—A course is given at the Medical School in San

(2) Author of the *Landmarks of botanical history*, Part I. Published in the *Smithsonian miscellaneous collection* (part of vol. 54, Washington, 1909). See note on GREENE by W. L. JEPSON in *Dictionary of American biography* (vol. 7, 564-5, 1931).

Francisco by SANFORD V. LARKEY, Assistant Professor of medical history and bibliography.

In addition to these courses it should be mentioned that Professor R. H. LOWIE, who is deeply interested in the history of science, discusses certain aspects of it in his course on *Anthropology* entitled Chapters in cultural history.

The Department of *Philosophy* offers a course on the Philosophy of science, given by Assistant Professor P. MARHENKE. It consists in an examination of contemporary philosophical interpretations of the results and theories of the natural sciences with special reference to the views of POINCARÉ, CASSIRER, and WHITEHEAD.

Finally the Department of *History* has cooperated with the other departments in promoting the history of science. The professor of mediaeval history, JAMES WESTFALL THOMPSON, (formerly in Chicago), is especially interested in the subject, as our readers well know (3). He has been preparing for years a general bibliography of it, and has taken considerable pains to increase the collections on the history of science in the library of the University of Chicago (4).

Thus far I have dealt only with separate sciences, but many efforts have been made since 1930 to introduce the teaching of the *history of science in general*. Prof. CHARLES SINGER of University College, London, arrived in the summer of that year to deliver a course on the history of biology, which led to the publication of his admirable textbook on the subject (5). Professor L. J. HENDERSON of Harvard University was invited to come for a semester in 1931 and began a general course on the history and philosophy of science, but illness obliged him to interrupt it and the lectures were continued by Prof. LENZEN and Prof. LARKEY. In 1932, Dr. SINGER returned and gave a general course on the history of science (the publication of which in bookform is in preparation). In the spring semester this course was listed under the Department of Zoology, but for the fall semester a

(3) J. W. THOMPSON: The introduction of Arabic science into Lorraine in the tenth century (*Isis*, 12, 184-93, 1929), a "bahnbrechend" paper which has been followed by various others by other authors in the same journal.

(4) M. LLEWELLYN RANEY: The University libraries. *University of Chicago Survey*, vol. 7, 225-8, 1933.

(5) Not yet reviewed in *Isis* because of various accidents. A review of it will soon appear.

new department was inaugurated called Department of the History of Science.

Professor SINGER's course was entitled " Lectures in the general field of the history of the physical and natural sciences." It may be worthwhile to add that it was open to juniors and older men and was available as a free elective in the upper division and as part of the major in the following departments : astronomy, botany, household science, paleontology, philosophy, and zoology in place of 3 elective units; bacteriology, as a substitute for civil engineering 124; physiology, as a substitute for zoology 117 B; physics, as a substitute for any 3 units allowed in courses in other departments.

In addition to this general course, Professor SINGER offered another one on " Scientific writings as literature," being assisted in this interesting project by members of the departments of English, Philosophy, and Medical History.

In the same year 1932, Professor HENRY E. SIGERIST, late Director of SUDHOFF's Institute in Leipzig and now Director of the WELCH Medical Library in Baltimore, gave a few lectures (see *Isis*, 18, 192). Finally my own appointment as Hitchcock Professor was a continuation of the same general purpose.

It is certain that but for the occurrence of the financial depression which curtailed considerably the university's resources, that new department would have been finally established on a solid and permanent foundation. I trust that it will be permitted to continue its existence in a humbler way until better times return. It would be unwise to lose the results of such long and patient efforts. By bringing together from time to time all the professors above mentioned and appointing one as secretary, it should be possible to continue the department without additional expenditure.

Two doctorandi are at present sponsored by the department, Miss SONOMA COOPER, who is preparing a thesis on the medical school of Montpellier in the Middle Ages (*Isis*, 15, 273) and THOMAS COWLES, who is studying the publications of the founders and early members of the Royal Society (6).

Dr. C. B. LIPMAN, Professor of plant physiology and Dean

(6) See his article on Dr. HENRY POWER, disciple of Sir THOMAS BROWNE, to appear in *Isis*.

of the Graduate Division, and Dr. G. D. LOUDERBACK, Professor of geology and Dean of the College, are both genuinely interested in our movement, and they both realize keenly that much more is involved than the history of science. It is clear that sooner or later colleges will have to provide a general scientific and cultural training for the many men and women who do not wish to become specialized technicians but rather to understand the scientific bearings of modern life. There is a growing need for such men and women, as educators, historians, philosophers, editors of scientific journals, administrators of scientific institutions, keepers of scientific museums and libraries, publicists, and statesmen. And it is also clear that the history of science would constitute an intrinsic part of such training. I have developed these views so often and in so many different ways that I need not insist upon them at present. It is more to the point to say that nowhere did I find a keener appreciation of them than at the University of California, and the friendly atmosphere which I enjoyed in Berkeley is partly due to the attitude of the two leading deans.

The library facilities of the University of California are already considerable and are rapidly increasing. In addition to the general library, there are a number of private libraries, notably the collection of anatomical, medical, and other scientific books made by Professor HERBERT MCLEAN EVANS (7), the large collection of biological and other scientific books made by Professor KOFOID, the smaller one made by Professor SETCHELL. To these resources must be added those readily available in the neighborhood, for example, the library of the California Academy of Sciences, and the Lane Medical Library, both in San Francisco. The Lane Medical Library is the best medical library of west of Chicago, and its historical department is remarkably strong. It even includes a good collection of medical books and Mss. in Arabic, Persian, and Turkish.

The Lane Medical Library is a part of Stanford University, located near Palo Alto, in another part of the Bay. There is some talk of organizing a medico-historical seminary which would

(7) Now for sale. I hope the University of California will be able to obtain possession of it.

be common to both universities, University of California and Stanford.

While I am mentioning Stanford, I may say that a course on the history of science is offered there by Professor F. O. KOENIG of the Chemistry Department, who was one of my own students.

The historical interest of the University of California community is also evidenced by the existence of clubs : the California Medical History Seminar meeting at irregular intervals at the Bohemian Club of San Francisco, and the Singer History of Science Club meeting in Berkeley. The former club is organized by CHAUNCEY D. LEAKE, and the latter by THOMAS COWLES and EDITH A. PICKARD.

The Bay of San Francisco is not the only center of interest in our studies in California. Another exists in the south, in Pasadena and San Marino. There may be found in close proximity three great institutions : the Henry E. Huntington Library, Art Gallery, and Botanic Garden, the Mount Wilson Observatory, and the California Institute of Technology. Each of these three institutions has some points of contact with us. The Huntington Library contains an exceedingly rich collection of books which will be particularly valuable for the history of English thought in the seventeenth and eighteenth centuries. Properly qualified students will find there perfect accomodations and excellent help. Thanks to ROBERT A. MILLIKAN's (8) generous humanism the " Cal. Tech. " will gradually devote more and more attention to our studies. Finally the Pasadena section of the Mount Wilson Observatory contains the nucleus of a fine collection of books on the history of astronomy. Among the astronomers attached to it, Dr. EDWIN P. HUBBLE seems to be especially interested in the history of his science. Nor should I forget Dr. GEORGE ELLERY HALE, the emeritus director of the Mount Wilson Observatory, who has a rich historical collection in his own private observatory, and has always had considerable historical curiosity, witness the fact that he was one of the first friends and patrons of *Isis*.

To these professional and official activities should be added the private activities—as scholars and writers—of these men and

(8) See his *Science and the New Civilization*, New York, 1930 (*Isis*, 14, 446-8).

of many others. To quote a single instance, Dr. H. HARRIS of San Francisco has recently published a *California's Medical Story*, which is a most delightful book of which we hope to publish a review before long.

Long as this account is, it is still very incomplete—for I ought to have said something of the endless courtesies extended to me far beyond my deserts and of the abundant hospitality which the good men and women of California showered upon me, though I had no claim to it, except my persistent devotion to the History of Science.

Harvard Library, 185. GEORGE SARTON.
Cambridge, Mass., Feb. 21, 1933.

APPENDIX

LIST OF SARTON'S CALIFORNIA LECTURES IN 1933.

Hitchcock Professorship

The Hitchcock Endowment Fund was established by the late Mr. CHARLES M. HITCHCOCK as an endowment for " a professorship in the University of California, for free lectures upon scientific and practical subjects but not for the advantage of any religious sect nor upon political subjects."

The Hitchcock Endowment Fund has been very much enlarged recently through the generous bequest of Mrs. LILLIE HITCHCOCK COIT, daughter of the late CHARLES M. HITCHCOCK, and the endowment is now known as the Charles M. and Martha Hitchcock Chair. With the increased income which now accrues from the endowment it has become possible for the University to enlarge the scope of work and extend the time for the visit here of the Hitchcock Professor, thus making possible much more intimate contact between him and the students and faculty members concerned. This carries with it the change of title from Hitchcock Lecturer to Hitchcock Professor for persons appointed under this foundation.

From 1909 to 1932 there have been altogether 22 Hitchcock Lecturers representing different branches of science, and three Hitchcock Professors : PETER DEBYE, J. B. S. HALDANE, and GEORGE SARTON.

SARTON's lectures were arranged as follows and delivered in Room 11, Wheeler Hall, 8 P.M. :

Medieval Science and Medieval Culture (with special reference to the Mediterranean World and the Arabic writings)

1. January 10 — The Unity and Diversity of the Mediterranean World.
2. January 12 — The Birth of Islām and the Development of the Dār al-Islām.
3. January 16 — The Golden Age of Arabic Culture.
4. January 18 — Some Remarks on the Arabic Language and Literature.
5. January 23 — The Arabic Tradition of Ancient Science.
6. January 30 — Arabic Contributions to the Revival of Ancient Science.
7. February 1 — The Decadence of Islām and its Causes.
8. February 6 — Medieval Thought and Modern Life.

A ninth lecture on "The Muslim synthesis of Greek and Jewish thought," to be placed between the 6th and 7th, was not delivered. We hope to have these lectures published in book form; they are still unwritten.

Other lectures (in Berkeley, except when otherwise stated)

10. Jan. 13 — Seminar on the Teaching of the history of science.
11. Jan. 15 — SIMON STEVIN (History of Science Club, Women's Faculty Club).
12. Jan. 18 — The study of nature in Islām (Seminar in biology).
13. Jan. 20 — Second seminar on the Teaching of the history of science.
14. Jan. 21 — On the slowness of human progress (California Medical History Seminar, Bohemian Club, San Francisco).
15. Jan. 26 — The History of Science and the New Humanism (The University of California at Los Angeles).
16. Jan. 27 — The meaning of mediaeval science (The Carnegie Institution of Washington, Mt. Wilson Observatory. The Athenaeum, Pasadena).
17. Jan. 31 — Magic, science, religion (Phi Beta Kappa chapter, University of California).
18. Jan. 31 — The rise and decline of Arabic science (Medical

School of the University of California, San Francisco).

19. Feb. 1 — Physical and chemical theories in the Middle Ages (Physical seminar).
20. Feb. 2 — The History of Science and the New Humanism (Stanford University, Palo Alto).
21. Feb. 3 — Seminar on the Study of the History of Science.
22. Feb. 3 — The occurrence and diffusion of inventions (Anthropological seminar at Prof. H. M. ADLER's residence).
23. Feb. 7 — Mediaeval science and its influence on modern science. Final plea for the study of the History of Science (San Francisco County Medical Society, San Francisco).

Was ist ägyptische Geometrie?

Die Frage nach dem Wesen der ägyptischen Mathematik hat für uns, die wir die Luft griechischer Wissenschaft atmen, nur dann Sinn, wenn sie bedeutet: Wie kann man den Geist ägyptischer mathematischer Betätigung gegen den Geist griechischer Mathematik abgrenzen? Die Erforscher der bildenden Kunst der Ägypter sind in einer glücklicheren Lage als die der Mathematik. Eine Fülle von Kunstwerken der verschiedenen Zeiten ist uns erhalten und hat es ermöglicht, eine klare Formulierung dessen zu versuchen, was ägyptische Kunst als vorgriechische Kunst scheidet von *der* griechischen Kunst, die in dem für die Entwicklung der Menschheit so bedeutungsvollen fünften vorchristlichen Jahrhundert entstand und von da an die Kunstbetätigung aller von ihr berührten Kulturen entscheidend beeinflusste. Wie spärlich sind, verglichen mit all den Bauwerken, Rund- und Flachbildern des Alten, Mittleren und Neuen Reiches die uns erhaltenen Überreste absichtlich mathematischer Arbeit der Ägypter! Noch dazu fallen sie, sofern sie vorgriechischer Zeit angehören, alle ins Mittlere Reich. Was vollends die *geometrische* Betätigung der Ägypter betrifft, auf die wir im Folgenden unser besonderes Augenmerk richten wollen, so sind da die Aufschlüsse der mathematischen Dokumente besonders karg, denn geometrische Entwicklungen enthalten sie nicht, sondern kaum mehr als numerische Berechnungsbeispiele, in denen nur neben nichtgeometrischen Gegenständen auch Flächen- und Rauminhalte, ferner Strecken und Streckenverhältnisse an einfachen Figuren und Körpern nach gewissen unbewiesenen und nicht formulierten Vorschriften ausgerechnet werden. Zu diesen unmittelbaren Zeugnissen ägyptischer Geometrie kommen allerdings die mittelbaren. Die Himmelskunde, die Vermessungskunst, die Baukunst, die seit ältester Zeit hochentwickelte Tischlerei, überhaupt die Technik im weitesten Sinne lassen Rückschlüsse auf

die geometrische Vorstellungsgabe und Gestaltungskraft der Ägypter zu, zum Teil auch auf ihre geometrischen Kenntnisse und den Grad der Logik oder Unlogik bei der geometrischen Betätigung. BORCHARDTS (1) Untersuchung der Sonnenuhren ist ein Beispiel dafür, welche Aufschlüsse so die ägyptische Technik über die ägyptische Geometrie zu geben vermag.

Unter den mittelbaren Quellen zur ägyptischen Geometrie verdient die *bildende Kunst* eine besondere Stelle. Wenn wir mit SCHÄFER (2) beim Kunstwerk die „naturerforschende Schicht" oder „Gerüstschicht" (3) von der „Ausdrucksschicht" unterscheiden, ohne zu behaupten, dass sich diese Scheidung stets scharf vollziehen lasse und dass mit ihr das allerletzte Wort gesprochen sei, und wenn wir im Lichte von SCHÄFERS Erkenntnissen in jener „Gerüstschicht" wesentliche Züge finden, die die vorgriechische Betätigung von der griechischen scheiden, so ist es von hohem Interesse, diese Gerüstschicht auf ihre geometrischen Inhalte und Methoden hin zu untersuchen und diese Inhalte und Methoden mit den entsprechenden der *griechischen* Kunst zu vergleichen.

Wie kann es aber gelingen, bei all der starren Formalität und verschlossenen Fremdheit, in der uns Ägyptisches zunächst entgegentritt, in den Geist, in die Seele ägyptischer Geometrie einzudringen? Die Untersuchung der ägyptischen Kunst fand wesentliche Aufhellung durch den Vergleich mit der Kunst solcher jetzt lebender Menschen, die noch nicht mit griechischer Kunst oder ihren Auswirkungen in Berührung kamen, insbesondere auch durch den Vergleich mit der Kunst des Kindes. Sollte uns nicht für das Verständnis der Geometrie der Ägypter des Mittleren Reiches die *Geometrie des Kindes* Fingerzeige geben können?

Ein junger Mathematiklehrer, der eben von der Hochschule kommt und in allen seinen Adern und Poren voll ist von den Begriffen und Methoden wissenschaftlicher (und das heisst zu

(1) LUDWIG BORCHARDT, Die altägyptische Zeitmessung. Berlin und Leipzig, 1920.

(2) HEINRICH SCHÄFER, Von ägyptischer Kunst. Dritte Auflage. Leipzig, 1930. S. 330.

(3) In letzter Zeit gebraucht Prof. SCHÄFER statt „naturerforschende Schicht" den Ausdruck „Gerüstschicht", da der erstere Ausdruck irreleitende Gedanken an „Naturalismus" nach sich ziehen könnte.

einem wesentlichen Teil griechischer) Mathematik, wird vor eine Klasse unverdorbener Zwölfjähriger gestellt, um sie für die wissenschaftliche Geometrie vorzubereiten. Er fordert, getreu dem Grundsatz, die Selbsttätigkeit zu wecken, die Schüler auf, *Dreiecke* an die Tafel zu zeichnen, und wundert sich, dass die Knaben ihm nur *gleichschenklige* Dreiecke bieten, und zwar schön stabil so gezeichnet, dass die Basis waagerecht und unten liegt (4). Manche seiner Zöglinge lassen sogar nur das *gleichseitige* Dreieck gelten. Fällt ein Dreieck durch das Ungeschick des Zeichners nicht recht symmetrisch aus, so wird es von einigen als „nicht richtig" oder „falsch", von anderen als „nicht schön" erklärt. Der Lehrer verlangt Vierecke und erhält nur Quadrate, allenfalls von einem ganz Schlauen auch ein Rechteck, alle natürlich in Grundstellung gezeichnet, d.h. so, dass ein Seitenpaar waagerecht, das andere senkrecht steht. Und wenn der junge Lehrer dann sagt, er wolle nun selbst einmal ein Viereck anzeichnen, und er zeichnet ein „beliebiges", ungleichseitiges und ungleichwinkliges, in „beliebiger" Lage, so erntet er wohl einen Heiterkeitserfolg. Diesen Knaben fehlt eben der „Begriff" des „beliebigen", „allgemeinen" Dreiecks und Vierecks. Sie lassen nur besondere, und zwar regelmässige oder symmetrische Dreiecke und Vierecke gelten. Woher haben sie diese Vorstellungen? (5) Die Umwelt, die sich der Mensch selbst geschaffen hat, ist bekanntlich reich an solchen regelmässigen, rechtwinkligen und symmetrischen Formen. Gebäude und ihre Teile und die mannigfachsten Gebrauchs- und Schönheitsgegenstände des Lebens weisen diese speziellen Formen auf. Dass der Mensch auf diese Formen verfiel, hat einmal physikalisch-technische Gründe. Die Natur selbst liefert den rechten Winkel, indem sie die Schwerelinie senkrecht zur Niveaufläche stellt. Das symmetrische Dach, die quaderförmige Mauer

(4) Vgl. ERNST GOLDBECK, Psychologische Erfahrungen im ersten geometrischen Unterricht. *Unterrichtsblätter für Mathematik und Naturwissenschaften* 37, 1931, 334-338. Diesen schönen Aufsatz las ich beim Aufzeichnen meiner verwandten Gedankengänge. GOLDBECK betont besonders die Rolle des *Ästhetischen* im vorwissenschaftlichen geometrischen Denken.

(5) Herr Professor H. EHELOLF in Berlin fragt, ob die Versuchsschüler schon Lehrbücher mit "Normalfiguren" gesehen haben. Die obigen Mitteilungen gründen sich auf gelegentliche, aber langjährige Beobachtingen, bei denen nicht versucht wurde, die Herkunst der Vorstellungen und Begriffe der Schüler experimentell nachzuforschen.

entsprechen den Gesetzen der Schwerkraft. Aber diese Formen befriedigen zugleich das Bedürfnis nach einfacher geometrischer *Schönheit*. Wir können es hier unentschieden lassen, ob diese geometrische Schönheit nur dasselbe ist, wie jene technische Zweckmässigkeit, oder aus derselben Wurzel entspringt. Das gleichschenklige Dreieck ist schöner als das ungleichseitige, und die unbewusste Freude an der Schönheit der Form, aus einem schöpferischen Quell im Menschen entspringend, gibt dem Knaben die Lust zur Herstellung solcher Formen. Auf das Vorhandensein des Lustgefühls ist Nachdruck zu legen. Der Lehrer sollte sich nicht über die mangelnde Logik des Kindes entrüsten, das sich unter einem „Viereck" nur ein rechtwinkliges Viereck oder gar ein Quadrat vorstellen will. Denn hier tritt uns „urwüchsige" Geometrie entgegen, die auch in der Terminologie der wissenschaftlichen Geometrie ihre unanslöschlichen Spuren hinterlassen hat, wenn diese z.B. das gleichseitig-rechtwinklige Viereck als τετράγωνον (6) oder quadratum bezeichnet. PESTALOZZI geht in seinem „ABC der Anschauung" vom Quadrat aus, HERBART (7) von der unendlichen Menge der ungleichseitigen Dreiecke. PESTALOZZI hat, wie ich meine, den tieferen pädagogischen Blick gezeigt. Die Analyse der „geometrischen" Vorstellungen des 12-jährigen Knaben ergibt also, dass diese stark durchsetzt sind von physikalisch-technischen Triebkräften (Standfestigkeit) und schönheitlichen Triebkräften (Symmetrie, Regelmässigkeit). Dagegen fehlen die höheren und abstrakteren Begriffe der „allgemeinen" Figuren.

Nun stellen wir uns einmal vor, unser junger Geometrielehrer träte vor eine Zuhörerschaft von Ägyptern des Mittleren Reiches, und zwar nicht vor eine Schülerklasse, sondern vor eine Gesellschaft von Mathematik*lehrern*. Dürfte er bei ihnen für den Begriff und die mathematische Behandlung des *allgemeinen* Dreiecks verständnisvolle Gesichter erwarten? Würden sie die Beschäftigung mit einem beliebigen, in beliebiger Lage gezeichneten Viereck als Gegenstand ihrer „Geometrie" gelten lassen, oder würden sie nur das *Rechteck* als die „vierige" *(ifd)* Figur anerkennen,

(6) EUKLIDS Elemente, Buch I, Definition 22.

(7) J. F. HERBART, Pestalozzi's Idee eines ABC der Anschauung usw. 2. Ausgabe 1804.

mit der man sich in der durch die Einleitungsworte des Papyrus Rhind angekündigten, zu den Geheimnissen der Dinge hinabsteigenden „Wissenschaft" befasst? Es ist gewagt, dass man seit EISENLOHRS Vorgang das Wort *śpd.t* immer durch „Dreieck" übersetzt. In diesem Wort, das man vielleicht mit „das Spitz" oder „die Spitze" wiedergeben könnte, steckt nichts von der Dreizahl der Ecken, und dass dieses Wort den allgemeinen Dreiecksbegriff bezeichnet, der uns von der griechischen Geometrie her geläufig ist, müsste erst bewiesen werden. PEET (8) und GUNN (9) erwägen bei Behandlung von Rh 51 (10) auch die Möglichkeit, dass hier ein allgemeines (ungleichseitiges) Dreieck gemeint sei. Dann müsste also die zugehörige Figur nicht die verkleinerte Abbildung eines durch die 2 gegebenen Stücke *bestimmten*, gleichschenkligen oder rechtwinkligen Dreiecks bedeuten, sondern die unendliche Menge aller verschieden gestalteten Dreiecke von gegebener Grundlinie und Höhe vertreten. Hier muss man fragen, ob die logische Geisteseinstellung der Ägypter dies zulässt. Von der Behandlung und verkleinerten Wiedergabe einer durch die gegebenen Stücke bestimmten Figur bis zur Behandlung und Zeichnung einer unbestimmten, unendlich viele Einzelfälle in sich schliessenden Figur ist ein weiter Schritt. Man bedenke, welche Rolle in der Philosophie PLATONS der Begriff des allgemeinen Dreiecks als Beispiel für einen Begriff uns als Vorstufe der Idee spielt. Man spürt, dass es sich hier um eine Errungenschaft der mathematischen und philosophischen Logik der Griechen handelt. Wir haben keine Anzeichen dafür, dass ägyptische Denker zu diesem Begriff emporgestiegen sind und imstande waren, in der Figur eines einzigen Dreiecks das Bild einer unendlichen Mannigfaltigkeit verschieden gestalteter Dreiecke mit dem geistigen Auge zu erblicken.

Der verdienstvolle Herausgeber des Moskauer Papyrus nimmt

(8) T. ERIC PEET, The Rhind Mathematical Papyrus. Liverpool 1923, S. 92. Bei dieser Gelegenheit sei die prächtige, auch dem Anfänger in der ägyptischen Sprache überaus nützliche Ausgabe des Papyrus Rhind durch CHACE und seine Mitarbeiter genannt. Die reichhaltige Bibliographie von ARCHIBALD verleiht der Ausgabe besonderen Wert. (CHACE, A. B., MANNING, H. P., BULL, L. S., and ARCHIBALD, R. C., The Rhind mathematical Papyrus. Oberlin (Ohio), 2 Bde, 1927 und 1929.)

(9) *JEA* (= *The Journal of Egyptian Archaeology*) XII, 1926, S. 133.

(10) Rh 51 bedeutet: Aufgabe 51 des Papyrus Rhind.

an, das Wort *ifd* drücke ein „Viereck *überhaupt*" aus (11), das heisst also doch wohl, ein *beliebiges* oder *allgemeines* Viereck; aber für diese Annahme fehlen alle Unterlagen. Da, wo das Wort *ifd* in der mathematischen Literatur vorkommt, ist immer ein rechtwinkliges Viereck gemeint. Das Wort kann auch, wie das Wörterbuch so schön sagt (12), einen „viereckigen Steinblock" bezeichnen, das heisst zweifellos, einen (von Rechtecken begrenzten) Quader. Dass *p.t* der Terminus für das Rechteck gewesen sei, ist eine unhaltbare Hypothese STRUVES (13).

Die in den bekannten mathematischen Schriften des Mittleren Reiches vorkommenden Figuren und Körper (14) sind bestimmte geometrische und stereometrische Gebilde einfacher Art, die sich durch das Vorhandensein rechter Winkel, oder durch Symmetrie oder durch Regelmässigkeit, meist durch mehrere dieser Eigenschaften zugleich auszeichnen. Jedenfalls trifft dies zu für all die Gebilde, über deren Deutung keine Zweifel bestehen. Dies sind : Quadrat, Rechteck, Kreis, Würfel, quadratische Säule, Quader, gerader Kreiszylinder, regelmässige vierseitige Pyramide, regelmässiger vierseitiger Pyramidenstumpf. Alle diese Gebilde sind in jeder Aufgabe, in der sie vorkommen, nach Gestalt und Grösse durch die gegebenen Stücke eindeutig bestimmt. Soweit aber die Deutung unsicher ist, wie bei dem Gebilde von Rh 60 (Kegel?) und dem „Korb" von M10 (15), spricht alles dafür, dass es sich auch hier um einfache, zentral- oder bilateral-symmetrische Gebilde handelt, deren Gestalt und Grösse durch die

---

(11) Mathematischer Papyrus des Staatlichen Museums der Schönen Künste in Moskau, herausgeg. u. kommentiert von W. W. STRUVE. *Qu. u. St. (= Quellen und Studien zur Geschichte der Mathematik)* A, I, Berlin 1930, S. 126.

(12) WB. I, 71. (= ADOLF ERMAN und HERMANN GRAPOW, Wörterbuch der ägyptischen Sprache, I, Leipzig 1925, S. 71). Haben die Verfasser des Wörterbuchs bewusst oder unbewusst die Übersetzung „viereckiger Steinblock" gewählt, die ein prächtiges Beispiel jetzt wie vor 4000 Jahren vorhandener geradansichtig-vorstelliger Betrachtungs- und Ausdrucksweise ist? Wenn heutzutage ein Schüler von einem „viereckigen Stein" spricht, so pflegt der boshafte Geometrielehrer ihn mit teuflischer Freude zu belehren, dass das ein Tetraeder sei. Vgl. übrigens die Bezeichnung *λίθος τετράγωνος* für einen Quader z. B. bei HERO, *Mensurae*, ed. HEIBERG, S. 166.

(13) Vgl. PEET, *JEA* 17, 1931, S. 160.

(14) Vgl. die Übersicht in O. NEUGEBAUER, Die Geometrie der ägyptischen mathematischen Texte. *Qu. u. St.* B, I, 413-451.

(15) M 10 bedeutet : Aufgabe 10 des Moskauer Mathematischen Papyrus.

Angabe von 1 oder 2 Massen völlig bestimmt ist. Von jedem der wenigen vorkommenden „Dreiecke" sind nur zwei Stücke gegeben. Wollen wir also nicht auf einmal bei den Dreiecken den Ägyptern den unerhört hohen Begriff einer allgemeinen, unendlich viele Einzelformen umfassenden Figur zuschreiben, so folgt schon hieraus, dass es sich um spezielle, rechtwinklige oder gleichschenklige Dreiecke handeln muss, was sich auch durch die Forschungen STRUVES (16) aus anderen Gründen bestätigt. Mit dem Dreieck von Rh 51 wird auch das Trapez von Rh 52 gleichschenklig. Soll dann die Figur von Rh 53 eine bestimmte Gestalt haben, so muss auch sie rechte Winkel enthalten oder symmetrisch sein.

Indem ich hier der vielumstrittenen Frage: „Hatten die Ägypter den Begriff des allgemeinen Bruches?" (17) die Frage an die Seite stelle: „Hatten die Ägypter den Begriff des allgemeinen Dreiecks?", bin ich auf Widerspruch gefasst. Wie oft, so wird man mir entgegenhalten können, mag schon ein Landmesser des mittleren Reiches in die Lage gekommen sein, ein *ungleichseitiges* Dreieck zu vermessen! Wenn er es dann durch eine Höhe in zwei rechtwinklige Dreiecke mit den Inhalten $\frac{1}{2}\ h.p$ und $\frac{1}{2}\ h.q$ zerlegte, wie nahe lag es da, bei immer wiederholter Erfahrung $\frac{1}{2}\ h.p + \frac{1}{2}\ h.q$ zu $\frac{1}{2}\ h.\ (p + q) = \frac{1}{2}\ h.c$ rechnerisch zusammenzufassen! Zugegeben, dass er diese praktische Beobachtung des distributiven Gesetzes machen konnte. Wir werden dem Ägypter später bei unserem Versuch, die Auffindung des Pyramidenstumpfinhaltes zu erklären, eine ähnliche Zusammenfassung zutrauen. Diese Zusammenfassung wäre aber zunächst nur eine erfahrungsmässige Vereinfachung der Rechnung und der Messung für den praktischen Landmesser. Von da bis zum Begriff des allgemeinen Dreiecks, das alle speziellen Dreiecke als Oberbegriff umfasst und ein bedeutungsvoller, zentraler Gegenstand geometri-

(16) STRUVES Argumentation S. 130 unter *e*) kann ich allerdings in dieser Form nicht anerkennen. Er macht stillschweigend die Voraussetzung, dass 3w und *šfnw Seiten* des Dreiecks sein müssen. Fasst man sie als Basis und Höhe eines gleichschenkligen Dreiecks auf, was *an sich* auch möglich wäre, so ist die Aufgabe ebenso lösbar.

(17) Hier seien nur zwei hervorragende Werke von Vertretern der in dieser Streitfrage einander gegenüberstehenden Parteien genannt: KURT VOGEL, Die Grundlagen der ägyptischen Arithmetik, München 1929. O. NEUGEBAUER, Arithmetik und Rechentechnik der Ägypter. *Qu. u. St.* B, I, S. 301-380.

scher Betätigung ist, scheint mir ein weiter Schritt zu sein (18).

Nie begegnen uns, wenn man meiner Ansicht über die „Dreiecke" zustimmt, schiefe und unregelmässige Figuren oder Körper, nie, um es mathematisch zu sagen, die *allgemeinen* affinen oder gar projektiven Transformationen der genannten rechtwinkligen und symmetrischen Gebilde. Wir haben zwar Rechtecke, nicht aber unsymmetrische Parallelogramme oder gar allgemeine Vierecke, wir haben rechtwinklige, nicht aber schiefwinklige Parallelflache, wir haben gerade regelmässige, nicht aber schiefe, unregelmässige Pyramiden. Es scheint, als ob derartige allgemeinere Gebilde überhaupt noch nicht in den Gesichtskreis des ägyptischen Geometers gerückt sind. Warum?

Man wird vielleicht antworten: Weil die Geometrie der Ägypter praktisch ist, und weil insbesondere z.B. die Baukunst aus statischen Gründen zuerst zu jenen geraden und symmetrischen Gebilden führte. Diese Begründung genügt aber nicht. Die Landvermessung führte doch auch auf *un*regelmässige Dreiecke und Vierecke, wie die späteren Felderberechnungen von *Edfu* wirklich zeigen. Die Beschäftigung mit der Geometrie dient zwar der Befriedigung praktischer Bedürfnisse, ist aber an sich selbst eine lustvolle schöpferische Phantasiebetätigung, bei der der Ausübende nicht an die Nützlichkeit zu denken braucht. Beim Betreiben der wissenschaftlichen Geometrie, die die Griechen begründet haben, bestehen die Lustmotive zu einem wesentlichen Teil in der intellektuellen Freude am logischen Aufbau und an der begrifflichen Allgemeinheit. Der vorgriechische Geometer, dem diese Freude versagt war, hat seine Lust an der schönen

(18) Eine babylonische Felderskizze, die ALLOTTE DE LA FUYE nach der Schrift einem der Zeit der Könige von *Ur* benachbarten Zeitbereich zurechnet, enthält für unser Auge u.a. zwei ungleichseitige Dreiecke, deren jedes durch eine Höhe in zwei rechtwinklige Dreiecke zerlegt ist. Der babylonische Landmesser sieht anscheinend nicht die ungleichseitigen Dreiecke, sondern nur ihre rechtwinkligen Teildreiecke, deren Flächeninhalte er einzeln ausrechnet und (zusammen mit einer fünften, rechteckigen Teilfläche) summiert. (ALLOTTE DE LA FUYE, Un cadastre de *Djokha*. *Revue d'Assyriologie* XII, 1915, S. 47-45)

Wenn man die schönen, regelmässigen Figuren der babylonischen Tafel BM 15285 anschaut, möchte man an die geometrische Erzeugung künstlerischer Flächenmuster denken. C. J. GADD hält es aber für klar, dass diese geometrischen Konstruktionen, die er der ersten babylonischen Dynastie zuweist, die Vermessung oder Parzellierung von Land erleichtern sollten. (C. J. GADD, Form and Colours. *Revue d'Assyriologie* XIX, 1922, S. 149-159.)

Geradheit, Symmetrie und Regelmässigkeit der Einzelform, die zugleich eine stabile Form für die Technik ist. Diese Lust ist ein Antrieb zur Geometrie, wenn wir die Herstellung und Berechnung solcher Einzelformen schon mit dem Namen „Geometrie" bezeichnen wollen. Gerade beim kunstbegabten Ägypter, der z.B. im Alten Reich seine Königsgräber als mächtige regelmässige vierseitige Pyramiden baute, der in Wandgemälden Decken- und Gewebemustern symmetrische mathematische Figuren verwandte(19), tritt diese ästhetische Triebfeder der Geometrie besonders hervor. Sie war übrigens auch später immer noch mächtig und ist es heute noch. Man denke an die Rolle, die die regulären Körper als Krönung der Euklidischen Bücher (20) spielen, ferner an die Verwendung dieser Körper für den Aufbau des Kosmos bei PLATO und bei KEPLER (21). Man denke an die Bedeutung, die Kreis und Kugel, diese schönen Sinnbilder der Regelmässigkeit, für die astronomischen Theorien der Griechen gewannen.

Ungeometrische Köpfe haben zu allen Zeiten einen heiligen Respekt vor der Geometrie aufbringen können, aber keine mitfühlende Freude an unserer *schönen* Wissenschaft. Wie ein begabtes Volk zur bildenden Kunst und zur Dichtkunst gekommen ist, das ist ihnen gar kein Problem. Wenn aber dasselbe begabte Volk seine geometrische Phantasie geregt hat, so sucht man nach Zwangsmotiven in der Notdurft des praktischen Lebens. So behaupten, angefangen mit HERODOT, viele Schriftsteller, der Nil habe durch seine Überschwemmungen der Felder die Ägypter zur Landmessung und damit zur Geometrie gewissermassen gezwungen. Aber ist das unsere schöne Geometrie, die da herauskommt, wenn noch die Landmesser von *Edfu* Dreiecke und Vierecke nach falschen Formeln berechnen? Ist es den Nilüberschwemmungen und der Notdurft des Alltags zu danken, wenn im Mittleren Reich die Böschung der Seitenwand einer regelmässigen Pyramide und der Rauminhalt eines regelmässigen Pyramidenstumpfs richtig berechnet wurden? Ist es nicht vielmehr der Kraftüberschuss einer glücklichen Zeit, der gleich-

(19) Hierauf wies schon CANTOR hin : CANTOR, Vorlesungen über Geschichte der Mathematik, I³, S. 107-108.

(20) Beachtenswert ist auch, dass die erste Aufgabe von EUKLIDS erstem Buche die Konstruktion eines *gleichseitigen* Dreiecks ist.

(21) Vgl. GOLDBECKS Ausführungen in dem unter (4) genannten Aufsatz.

zeitig die Phantasie des Baumeisters, des Künstlers und des Geometers zur Erschaffung und Berechnung der schönen Form in Bewegung setzt, unbeschadet der Tatsache, dass die Berechnungen des Mathematikers ihren unmittelbaren Anstoss in der Notwendigkeit für die praktische Bauausführung fanden (22) ? Erstes geometrisches Denken entflammt sich im Menschen wohl eher an den von ihm selbst erzeugten Gegenständen seiner "Kunst" (dieses Wort im alten, auch das Handwerk umfassenden Sinn genommen), als an unregelmässigen Gegenständen der empirischen Natur. Letztere gestaltet er um, indem er die seinem eigenen Geist entspringenden einfachen und regelmässigen Formen in sie hineinlegt.

Inwieweit neben den rein ästhetischen Motiven den schönen einfachen Figuren und Körpern, wie den niedrigen Zahlen, auch *mystische* oder *magische* Motive innewohnen, soll hier nicht untersucht werden. Auch treten wir den Zusammenhängen von Geometrie und *Kult* nicht näher.

Wenn wir in der „Geometrie" der Ägypter Kunstmotive wirksam zu sehen glauben, so ist es wichtig, umgekehrt in ihrer Kunst das Geometrische festzustellen.

Zunächst „lässt sich der jedem sofort auffallende „geometrisierende" Einschlag der ägyptischen Kunst bis in die rein konstruktive Anlage der Zeichnungen hinein verfolgen, teils an bestimmt angeordneten Hilfslinien, teils sogar an ganzen Koordinatennetzen, in die die Figuren eingetragen werden. Dies gilt sowohl für Menschen- und Tierdarstellungen, wie auch für reine Konstruktionszeichnungen der Bautechnik (Grundrisse, Pläne und Bauzeichnungen in Originalgrösse usw.)" Dies sagt NEUGEBAUER (14), der für die tiefere Analyse des geometrischen Einschlages der ägyptischen Kunst auf SCHÄFERS oben genanntes Buch verweist. Es kommt uns nun darauf an, die geometrische Eigentümlichkeit der „Gerüstschicht" ägyptischer Kunst festzu-

(22) Dass der Ägypter selber in seiner Mathematik etwas über das Alltägliche Hinausgehobenes sah, glaube ich schon in der Ankündigung auf der Titelseite des Papyrus Rhind zu erkennen. Argumente für den über das rein Praktische hinausgehenden Charakter der ägyptischen „Wissenschaft" brachte H. WIELEITNER in dem schönen Aufsatz : „War die Wissenschaft der alten Ägypter wirklich nur praktisch?" *Isis* IX, 1927, S. 11-28. Unter den neueren Verfechtern der „Wissenschaftlichkeit" der ägyptischen Mathematik seien ABEL REY und KURT VOGEL genannt.

stellen und zu vergleichen mit dem, was sich bei den Griechen an der betreffenden Stelle findet.

Hier müssen wir einen Blick auf den Zusammenhang zwischen der Malerei und der Geometrie der Griechen werfen. Es ist sehr bedeutsam, dass in demselben fünften Jahrhundert vor Christo, in welchem die griechischen Künstler die Darstellung der Schatten und die Perspektive erfanden, auch die wissenschaftliche Geometrie geboren wurde. Beide Entwicklungen standen in Wechselwirkung, so wenig wir auch im Einzelnen darüber bei der Lückenhaftigkeit der Quellen wissen. Beide Entwicklungen müssen in ihrer tiefsten Wurzel miteinander verwachsen sein. Nach VITRUV soll der Maler AGATHARCHOS für AISCHYLOS eine Skene gemacht und eine Schrift darüber hinterlassen haben. Dadurch angeregt hätten ANAXAGORAS und DEMOKRIT, also zwei führende Männer der Philosophie und — jedenfalls gilt das für den zweiten der beiden — der Mathematik, über Perspektive, und zwar im Besonderen über perspektivische Architekturmalerei, geschrieben. PAMPHILOS, ein Maler des 4. Jahrhunderts, genoss den Ruhm, als erster Maler in allen Wissenschaften bewandert gewesen zu sein. Er bezeichnete die Kenntnis von Arithmetik und Geometrie als unentbehrlich für die Malerei und hat eine Schrift *Περὶ γραφῆς* verfasst. „Die theoretische Begründung seiner Lehre hat ihn gewiss weit in Mathematik, Optik, Farbenlehre hinein geführt; er wird alles, was die Wissenschaft seiner Zeit von Proportionen, Farben, Perspektive, Schattenlehre und was immer sonst noch bot, herangezogen haben" (23). Die Optik des EUKLID legt Zeugnis davon ab, wie die Griechen die Perspektive mathematisch behandelten.

Eine rechteckige Wand, die von der Sonne beschienen wird, wirft auf eine ebene Fläche einen Schlagschatten von der Form eines Parallelogramms, das seine Form mit wechselndem Sonnenstand stetig ändert. So liefert die Natur selbst die affine Abbildung des Rechtecks. Die griechischen Künstler beobachteten den Schatten des Rechtecks und stellten ihn dar, und die griechischen Geometer schufen den Begriff des allgemeinen Parallelogramms und untersuchten seine Eigenschaften. Mag uns die griechische Geometrie auch unter dem Einfluss der platonischen Philosophie

(23) ERNST PFUHL, Malerei und Zeichnung der Griechen. München 1923.

in reiner, von der sinnlichen Beobachtung abgewandter Form entgegentreten, bei ihrer Entstehung und Entfaltung müssen sinnliche Beobachtungen mitgewirkt haben. Sind doch für PLATO selbst die Sinnendinge die Wecker der Ideen.

Abbildungen allgemeinerer Art als die parallelen Lichtstrahlen sie auf ebener Fläche erzeugen, liefert die Perspektive. Sie ergibt projektive Abbildungen. Ein Rechteck z.B. erscheint in der Perspektive im allgemeinen als Viereck mit nicht parallelen Seiten.

Es ist hier nicht unsere Aufgabe, zu untersuchen, wie weit die Griechen die mathematischen Gesetze der projektiven Abbildung erkannt haben. Eins aber ist sicher: Die Schatten und die perspektivischen Bilder, die PLATO noch als Schein und Trug verdammte, erschlossen dem griechischen Maler die Welt der affinen und projektiven Abbildungen „gerader" Figuren, sie lieferten allgemeine Dreiecke, schiefe Parallelogramme und allgemeine Vierecke, und eben diese Gebilde in ihrer Allgemeinheit sind es, die auch der griechische Geometer in demselben Jahrhundert untersuchte. So erhaben auch EUKLID in seiner *Geometrie* über das Sinnliche hinwegsieht, in seiner *Optik* blickt er trotz des streng geometrischen Aufbaues auf die Beobachtung des Malers, wenn er z.B. in Satz 36 sagt: „Die Räder der Wagen erscheinen zuweilen kreisförmig, zuweilen verzerrt (*παρεσπασμένοι*)". In streng geometrischer Weise untersucht er, von welchen Punkten aus betrachtet das Rad (der Kreis) als Kreis erscheint, und von welcher in verzerrter Form. Hier wird uns greifbar: eine praktische Aufgabe der Malerei, die perspektivische Darstellung eines Rades, führt den Geometer auf einen Weg, der in die allgemeine Untersuchung der projektiven Abbildungen des Kreises, in die Untersuchung der Kegelschnitte ausmünden muss: Die Sinnendinge Wecker der Ideen.

Um Missverständnisse zu vermeiden: Es wird nicht behauptet, dass nur des Malers Beobachtung des Rechtecksschattens den Geometer zu den Parallelogrammen führte, nur des Malers Rechtecksdarstellungen den Geometer zu den allgemeinen Vierecken, dass also etwa die Malerperspektive der Geometrie des allgemeinen Dreiecks und Vierecks vorhergehen musste. Aber ich glaube: Dieselbe verborgene Kraft, die bei den Künstlern den Durchbruch zu der Schattendarstellung und zu den allgemeinen projektiven Bildern der Dinge vollzog, brachte auch in

der griechischen Geometrie den Durchbruch von den „geraden", symmetrischen, rechtwinkligen Figuren zu den allgemeinen (24).

Das Wirken dieser Kraft wird uns aber erst besser bewusst, wenn wir über das Sinnlich- und Räumlich-Anschauliche hinaus ihre Verwurzelung und ihre Auswirkungen im *Logischen* betrachten. Der Grieche hat den *wissenschaftlichen Begriff* des allgemeinen Dreiecks, des allgemeinen Vierecks gebildet, der die Begriffe der besonderen Dreiecke, der besonderen Vierecke, wie Parallelogramm, Rechteck usw. unter sich umfasst, und er ist sich dieser Begriffsbildungen und -einteilungen bewusst geworden, denn er hat den Begriff entdeckt. Die ganze Geometrie ist für ihn schliesslich das geworden, was sie für uns noch ist: ein auf wenige Grundbegriffe und Grundsätze gegründetes, beweisbares logisches System. Philosophie und Mathematik haben sich in Wechselwirkung befruchtet, und ohne diese logische Vertiefung ist letzten Endes auch z.B. die Ausbildung der perspektivischen Praxis des Malers nicht denkbar.

Kein ägyptischer Künstler hat je den parallelogrammförmigen Schatten darzustellen versucht, den eine rechteckige Wand auf eine ebene Fläche wirft, keiner hat, von geringfügigen Ansätzen abgesehen, einen schräg gesehenen Gegenstand perspektivisch dargestellt. Ist es da wahrscheinlich, dass zu gleicher Zeit ägyptische Geometer das Dreieck und das Viereck in ihren allgemeinsten Formwandlungen untersuchten? Weniger noch, als es heute der Fall ist, traten in der Umwelt des Ägypters ungleich-

---

(24) Bemerkenswert erscheint mir, dass nach DIODOR ein verhältnismässig alter Gewährsmann, CALLIMACHUS (etwa 250 v. Chr.) von dem Phrygier *Euphorbos*, mit dem nach DIODOR *Pythagoras* gemeint ist, schrieb: Er *„war der erste der Menschen, der sogar ungleichseitige Dreiecke konstruierte und einen Kreis..."* (DIELS, Die Fragmente der Vorsokratiker, I³, Berlin 1912, S. 6-7). Hier an ungleichseitige *rechtwinklige* Dreiecke zu denken, wie es HEATH wohl tun möchte, scheint mir nicht hinreichend begründet. (Vgl. Sir THOMAS HEATH, A History of Greek Mathematics, I, Oxford 1921, S. 142).

Für den Übergang von den schönen symmetrischen Figuren zu ihren affinen Abbildungen gibt auch der griechische Terminus für das Parallelogramm einen Fingerzeig. Bei den Definitionen gebraucht EUKLID für diese Figur noch die Bezeichnung ῥομβοειδές, die ihre unmittelbare Verwandtschaft mit dem Namen des schönen, symmetrischen Parallelogramms verrät, des ῥόμβος (eigentlich „Kreisel"; wir nennen die Figur „Raute"). Später, beim Aufbau der Lehre, benutzt er für jene Figur die nüchterne, aber wissenschaftlichere Bezeichnung παραλληλόγραμμον.

seitige Dreiecke, schiefwinklige Parallelogramme und allgemeine Vierecke als interessebetonte Dinge aus der Fülle der symmetrischen Dreiecke und der Rechtecke hervor, die zugleich schönheitsbetont waren.

Nach SCHÄFER ist die „Gerüstschicht" der ägyptischen, wie aller vorgriechischen Kunst gekennzeichnet durch das „geradvorstellige Zeichnen" (25) und die „richtungsgerade Plastik". Haben die Gegenstände der Darstellung ausgezeichnete, aufeinander senkrechte Ansichtsflächen, wie der menschliche Körper oder ein Gebäude, so macht der Künstler sich von diesen Gegenständen oder ihren Teilen „geradeVorstellungen", d. h. er stellt sich z.B. einen Schultergürtel nur von vorn gesehen vor, einen Kopf nur von der Seite, jedes Gebilde in seiner eindruckvollsten Ansicht derart, dass parallele projizierende Strahlen senkrecht auf die betreffende Hauptansichtsfläche fallen. Aus solchen geraden Teilvorstellungen setzt nun der Flachbildner das Gesamtbild zusammen. Von unserem Standpunkt aus würden wir sagen, er füge die Teilvorstellungen *sprunghaft* (diskontinuierlich) aneinander. Neben die rechteckige Vorderfront eines Hauses zeichnet er die Seitenfront unbekümmert auch als Rechteck in seiner „wahren" Gestalt. Die Sprunghaftigkeit besteht darin, dass er für die Seitenfläche nicht mehr die für die Vorderfläche verwandten projizierenden Strahlen gebraucht, sondern neue, die im rechten Winkel zu den früheren stehen. So sind auch die Darstellungen der Teile des menschlichen Körpers diskontinuierlich aneinandergefügt. Die Bäume, die einen in seiner wahren, rechteckigen Form dargestellten Teich umgeben, werden, wie wir heute sagen würden, „in die Grundrissebene umgeklappt" dargestellt; aber damit soll nicht gesagt sein, dass auch der Ägypter sich eine solche Umklappung vorstellte. Ein symmetrisches Gebilde, z.B. ein gleichschenkliges Dreieck, wird auch im Bilde nur als solches erscheinen, ein rechter Winkel immer als rechter Winkel. Der Künstler hat noch kein Auge für die unsymmetrischen, schiefen Bilder der „geraden", „schönen" Figuren.

Und wie der Flachbildner noch nicht die beliebige Ansicht

(25) Statt des früher gebrauchten Ausdrucks „geradansichtig-vorstelliges Zeichnen" sagt SCHAEFER heute „geradvorstelliges Zeichnen", statt von „geradansichtiger Vorstellungsbildung" spricht er von „gerader Vorstellungsbildung", um das Wort „Ansicht" in diesem Zusammenhang zu vermeiden.

in Zentralperspektive erobert hat, so nimmt der Rundbildner zu seinem Werk nur bevorzugte, zu einander senkrechte Blickrichtungen auf die Hauptansichtsflächen ein, und die ganze Haltung der Teile des dargestellten menschlichen oder tierischen Körpers bleibt den Ebenen dieser Hauptansichtsflächen angepasst, um nicht zu sagen ange*presst*, wie die Hieroglyphen der Form des Rechtecks. Im Gegensatz dazu hat die griechische Plastik die Fülle der unendlich vielen Richtungen des Raumes erobert.

Die Gegenstände, die die künstlerische und die geometrische Phantasie des Ägypters erfüllen, sind eben als Ganzes und in ihren Teilen gebunden an das natürliche rechtwinklige Koordinatensystem, das festgelegt ist durch die natürliche Horizontebene (26), die erste Hauptansichtsebene des Gebildes und seine zweite, auf der ersten senkrecht stehende Hauptansichtsebene. Wie schwer es manchen Menschen fällt, sich in Anschauung und Denken von diesem Koordinatensystem der Natur zu lösen, erfährt jeder, der nicht vorgebildeten Menschen, z.B. Kindern, etwa die Konstruktion des Lotfällens von einem Punkte auf eine beliebig gezeichnete Gerade aufgibt, nachdem er ihnen die Lösung für eine wagerecht gezeichnete Gerade gezeigt hat.

Freilich ist in der ägyptischen Geometrie nicht alles rechtwinklig, ebensowenig wie in der ägyptischen Kunst. Aber wo das Schräge auftritt, ist es ein einzelnes Schräges, das unmittelbar an das Rechtwinklige ansetzt, wie ein schräger Unterarm sich an einen senkrechten Oberarm anschliesst. Selten tritt in der Plastik das Schräge aus einer der Hauptansichtsebenen heraus. Ähnlich in der Stereometrie : Die Böschung der Seitenfläche einer Pyramide ist als *śkd*, d.h. als Quotient der Katheten des Neigungsdreiecks ausgedrückt und dadurch sofort wieder auf die naturgegebene Horizontale und Vertikale bezogen. Bei dieser Höhe der theoretischen und praktischen Erkenntnis des Schiefen liegt wohl kein Bedürfnis vor, die Schrägheit anders als durch einen solchen Quotienten zu messen. Damit steht im Einklang, dass die Ägypter des Mittleren Reiches den Begriff und die Messung eines beliebigen

---

(26) Als Achsen in der Horizontebene wurden vielfach, z.B. bei Kultbauten von rechteckigem Grundriss, von den verschiedensten Völkern die Nord- Süd- und die West-Ostrichtung gewählt.

Winkels und seine Verwendung als selbständiges geometrisches Element nicht gekannt zu haben scheinen (27).

Wenn wir von unserem Standpunkt aus die Natur eines solchen Vorstellens, Anschauens und Denkens „verstehen" wollen, so werden wir schliesslich auch bei den Dingen der Kunstdarstellung und der geometrischen Figuren, Dingen, die zunächst der *Anschauung* anzugehören scheinen, erkennen, dass der Unterschied zu unserer modernen Behandlung damit zusammenhängt, dass das *Logische* sich auf einer anderen Stufe der Ausbildung befindet. Uns, die wir unmittelbar oder mittelbar der Begriffsscheidekunst der Griechen teilhaftig wurden, empört es, wenn in einem mathematischen Lehrbuch von einer (nach der Figur dreieckigen) Feldspitze die Rede ist, ohne dass uns überhaupt gesagt wird, ob es sich um ein rechtwinkliges oder ein gleichschenkliges oder was sonst für ein Dreieck handelt, und wenn dabei noch eine unglückselige, zwei- oder mehrdeutige Figur die Unsicherheit erhöht. Natürlich konnte der Ägypter einen runden von einem „viereckigen" Kornspeicher unterscheiden, aber zu einer den Begriff ausschöpfenden Einteilung etwa der Vierecke in Quadrat, Rechteck, Rhombus, Rhomboid und unregelmässiges Viereck, wie sie uns an der Spitze von Euklids erstem Buch begegnet (28), wird er sich nicht erhoben haben, und so wird er vor allem einen solchen rein logischen Begriff wie den des „*Vierecks überhaupt*", als des Begriffs, der alle diese besonderen Viereckbegriffe umfasst, nicht gebildet haben. Dass er sich irgend ein naturgegebenes unregelmässiges viereckiges Feld vorstellen konnte, möglicherweise auch schon im Mittleren Reich seine Fläche nach einer näherungsweise richtigen empirischen Faustregel zu berechnen pflegte, braucht darum nicht unmöglich zu sein. Aber als Geometer sah er wohl in einem solchen unregelmässigen Viereck etwas sehr Gleichgültiges und Hässliches, das nicht in seine „Wissenschaft" hineingehörte.

Man halte gegen die griechische Einteilung der Vierecke eine indische, die wir bei einem Scholiasten des BRAHMAGUPTA (29)

(27) Vgl. SOLOMON GANDZ, The origin of angle-geometry. *Isis*, XII, 1929, S. 452-481.

(28) EUKLIDS *Elemente*, Buch I, Deff. 21 und 22.

(29) Vgl. die englische Übersetzung : H. TH. COLEBROOKE, Algebra, with Arithmetic and Mensuration, from the Sanscrit of BRAHMAGUPTA and BHASCARA. London 1817, S. 295.

finden. Dieser Scholiast kennt fünf Gattungen von Vierecken. Es sind nach den Zahlenbeispielen, die der Scholiast selber gibt: Quadrat, Rechteck, gleichschenkliges Paralleltrapez mit nur zwei gleichen Seiten, gleichschenkliges Paralleltrapez mit drei gleichen Seiten und als letzte Gattung ein Sehnenviereck, das aus vier paarweise ähnlichen, rationalen rechtwinkligen Dreiecken zusammengesetzt ist. CANTOR (30) sagt, dass diese fünf Gattungen keineswegs „richtig" gewählt, sondern „unberechtigt" seien. „Sie erschöpfen den Begriff des Vierecks durchaus nicht". Einem Ägypter des Mittleren Reiches hätte die Aufzählung dieser Formen, mit Ausnahme der fünften, schwierigeren, sehr einleuchtend erscheinen müssen. Denn die vier ersten sind schöne, symmetrische Figuren, die fünfte hat eine kompliziertere Art arithmetisch-geometrischer Schönheit. „Nur mit ihnen hatte man sich zu beschäftigen". Ob hier bei den Indern uraltes Gut vorliegt, oder ob neue Wellen eines primitiveren Denkens griechische Wissenschaftsbegriffe überflutet haben, in jedem Falle treten uns hier Züge dessen entgegen, was wir als „vorgriechische Geometrie" zu kennzeichnen versuchen.

Die geometrischen Vorstellungen des Ägypters sind vielfach noch nicht gereinigt von ausserlogischen (empirischen, physikalischen, ästhetischen, vielleicht auch mystischen) Bestandteilen und daher noch nicht zu reinen Begriffen erhoben. Die Fähigkeit des *Abstrahierens* und des *Bildens allgemeinerer und höherer Begriffe* ist nicht in dem Grade, vielleicht nicht einmal in der Art vorhanden, wie wir sie bei den Griechen bewundern.

Wie merkwürdig mutet es uns zunächst an, dass der Ägypter nur mit Stammbrüchen und dem Bruch 2/3 rechnet. Aber auch hier muss man sich darüber klar werden, dass die Schöpfung des Begriffs des allgemeinen Bruches eine Tat ist, die aus einer höheren synthetischen Verstandestätigkeit entspringt. Es ist etwas anderes, ob ich mir 5/7 als die Summe von 5 einzelnen Siebenteln vorstelle — das konnte auch der Ägypter, — oder ob ich den durch ein Zahlenpaar angegebenen allgemeinen Bruch als *eine einzige Zahl* auffasse, mit der nach bestimmten Vorschriften gerechnet wird (17). Die Ägypter hatten noch nicht über Einheit und Vielheit philosophiert. Konnte der ägyptische Geist die

30) CANTOR, I³, S. 651.

Synthesis zu einem abstrakten Begriff, wie dem der „Schönheit", schon so vollzogen haben, wie unser Geist es tut? Es hatte noch kein *Plato* über die Idee des Schönen und über ihr Verhältnis zu der Vielheit der schönen Dinge geschrieben. Die Ägypter sprechen immer nur von der Schönheit eines bestimmten Menschen oder Dinges, nicht aber von der Schönheit *überhaupt*.

Da, wo *wir* aus einer höheren mathematisch-logischen Einheitlichkeit heraus ein einziges funktionales Gebilde schaffen, wie es die projektive Abbildung aller Gegenstände eines Raumes aus einem einzigen Augenpunkt ist, erscheint beim Ägypter das Gebilde in Teilgebilde zerfällt, die sprunghaft (diskontinuierlich) aneinanderstossen. Diese geometrische Sprunghaftigkeit äussert sich nicht nur in dem Nebeneinanderstellen von Teilbildern, die aus zu einander senkrechten Blickrichtungen gesehen sind, sondern auch z.B. darin, dass verschiedene Teile desselben Bildes, wie z.B. die Darstellungen eines Herrn und seiner Diener, in verschiedenem Massstabe erscheinen. Auf Kosten der uns geläufigen umfassenderen objektiven Naturdarstellung nach einheitlichen mathematisch-physikalischen Grundsätzen werden Vorstellungen der Seele unmittelbarer durch eine Diskontinuität wiedergegeben.

Nicht nur in der Kunst, sondern auch bei dem Versuch, raumzeitliche Vorgänge der *Natur* beschreibend zu praktischen Zwecken —zu diesen gehören auch die religiösen und magischen — wiederzugeben, zerlegt der vorgriechische Geist Zusammenhänge, die *wir* einheitlich durch eine kompliziertere funktionale Beziehung darstellen, in einfache, sprunghaft aneinandergereihte Einzelfunktionen. An solchen Einzelfunktionen hat er nur einen spärlichen Vorrat, zu dem das Gesetz der arithmetischen und das der geometrischen Reihe gehören. Ein schönes Beispiel hierzu bietet die babylonische Astronomie. Wenn wir heute die Grösse des erleuchteten Teils der Mondscheibe, so wie sie uns als ebene Fläche erscheint, vom Neumond bis zum Vollmond als Funktion der Zeit darstellen wollen, so werden wir in erster Annäherung, abgesehen von einem konstanten Faktor, die Funktion $1 - \cos \alpha$ von $\alpha = 0$ bis $\alpha = 90^0$ ansetzen, denn diese Fläche wächst proportional der Breite, die die Sichel in der Mitte besitzt. Ein babylonisches Mondtäfelchen (K 90) stellt die Zu- und Abnahme des erleuchteten Teils der Mondscheibe für die 30 Tage des

Monats durch Zahlen dar, wenn die zuerst von HINCKS gegebene, später von KUGLER (31) bekräftigte Deutung (32) richtig ist. Die Tafel setzt für die ersten fünf Tage eine geometrische, dann mit krasser Unstetigkeit des Gesetzes für die folgenden 10 Tage eine arithmetische Reihe an. In Fig. 1 habe ich diese Wertefolge für die erste Monatshälfte graphisch dargestellt und zum Vergleich

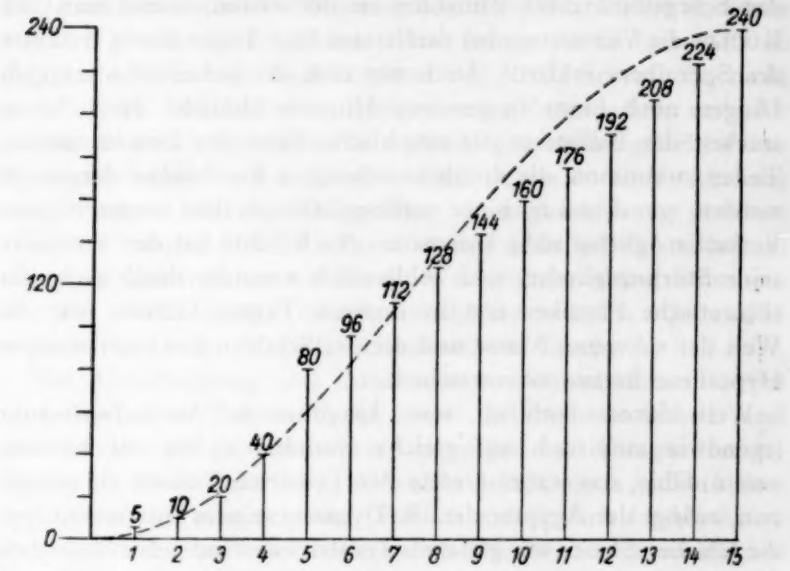


FIG. 1

die Kurve der modernen Funktion eingezeichnet. *Beobachtet* in unserem Sinne hat der Babylonier seine Werte natürlich keineswegs. Nur für den fünften Tag, der nach babylonischer Auffassung den ersten Teil des Monats abschloss, mögen nach KUGLER Schätzungen jener Sichelbreite vorgenommen worden sein. Es ist bemerkenswert, dass für diesen Tag die Sichelbreite mit dem runden Wert 1/3 des Durchmessers angesetzt ist, und KUGLER sagt, dass auch dieser Wert vielleicht schon vorher „angenommen" wurde. Wahrgenommen haben wird man ferner das beschleunigte Anwachsen der Sichelbreite an den ersten fünf Tagen, das zum

(31) FRANZ XAVER KUGLER, Sternkunde und Sterndienst in Babel. Münster. II, 1909/10, S. 45 und Ergänzungen, 1913, S. 102.

(32) Nach einer freundliche Mitteilung von Prof. NEUGEBAUER darf diese Deutung noch heute als richtig betrachtet werden.

Ansetzen der geometrischen Reihe führte, während für die folgenden 10 Tage die arithmetische Reihe den Wahrnehmungen und Zwecken genügte. Gar nicht zum Ausdruck gekommen ist hierbei, dass beim Rückwärtsdurchlaufen dieser 15 Tage der ergänzende dunkle Teil dieselben 15 Werte durchläuft. Dagegen ist die Abnahme der Erscheinung in der zweiten Monatshälfte das Spiegelbild ihrer Zunahme in der ersten, wenn man mit KUGLER die Verwirrung bei den letzten fünf Tagen durch Irrtümer des Schreibers erklärt. Auch wir arbeiten ja bei schwierigeren Dingen noch heute in gewisser Hinsicht ähnlich. Noch heute stückelt der Ballistiker die empirische Bahn des Geschosses aus Teilen zusammen, die durch verschiedene Funktionen dargestellt werden, von denen man nur verlangt, dass sie dem raumzeitlichen Verlauf möglichst nahe kommen. Noch heute hat der Astronom seine Störungsglieder, und schliesslich versucht doch auch der theoretische Physiker erst in unseren Tagen, Gebiete wie die Welt der schweren Masse und die der Strahlen aus einer *einzigen* Hypothese heraus zu verstehen.

Weit davon entfernt, eine kegelförmige Auslaufwasseruhr irgendwie empirisch auf gleiche Stunden aichen zu können, und unfähig, das wahre Gesetz der Teilstrichabstände zu errechnen, zwingt der Ägypter der 18. Dynastie seinem Instrument (33) die falschen Skalen mit gleichen Teilstrichabständen für denselben Monat und gleichförmig wachsenden oder abnehmenden Abständen von Monat zu Monat auf, weil die einfache funktionale Vorstellung der arithmetischen Reihe mit ihrer *schönen* Regelmässigkeit in seinem Geiste lebendig ist. So versuchten noch die Pythagoreer die Planetenabstände nach den schönen arithmetischen Gesetzen der Harmonie der Sphären zu regeln; so hat ein KEPLER noch vor etwa 300 Jahren den Planeten das Abstandsgesetz der ineinander geschachtelten regulären Körper aufzwingen wollen, deren Schönheit seinen Geist erfüllte. Bei den Sonnenuhren mit waagerechter Auffangefläche, deren älteste uns erhaltene der Zeit Thutmosis des Dritten, also der ersten Hälfte des 15. Jahrhunderts angehört, wählt der Ägypter, solange es eben geht, nämlich für den Zeitverlauf zwischen dem Ende der sechsten und dem Anfang der zweiten Stunde, entgegen aller astronomisch-trigono-

(33) Über ägyptische Wasser- und Sonnenuhren vergleiche man das unter (1) angeführte Werk von BORCHARDT.

metrischen Wirklichkeit, die ihm verborgen ist, Teilstrichabstände nach dem schönen Gesetz der arithmetischen Reihe. Dann kommt ein jäher Sprung : Der Teilstrichabstand der ersten Stunde wird unendlich gross. Auch bei den Treppensonnenuhren begegnet uns die gleichförmige arithmetische Stufung, während bei den Uhren mit senkrechter west-östlich gerichteter Auffangefläche und darauf senkrecht stehendem Schattenstab, die Stundenstrahlen *gleiche Winkel* bilden. Also immer das Einfachste. Den Ägypter scheint es, wie BORCHARDT mit Recht hervorhebt, nicht anzufechten, dass die beiden letztgenannten Sonnenuhren einander geometrisch widersprechen, da doch ein gleichwinkliges Strahlenbüschel verschieden von einem solchen ist, das durch Verbindung eines Punktes mit den Teilpunkten einer gleichförmig geteilten Geraden entsteht. Auch scheint er sich nicht gefragt zu haben, ob seine Sonnenuhren mit seinen Wasseruhren übereinstimmen, was *wir* durch einen Versuch entscheiden würden. Jede Uhr steht für sich da : einfach, schön und — falsch.

Die Untersuchung des Moskauer mathematischen Papyrus entfachte neue Erörterungen über die mathematischen und insbesondere die geometrischen Leistungen und Fähigkeiten der Ägypter. Diese habe man, so wurde gesagt, doch wohl bisher meist zu niedrig eingeschätzt. Freilich enthält auch der Moskauer Papyrus keine Spur der Kennzeichen griechischer geometrischer Schriften : Grundbegriffe. Axiome, Lehrsätze, Beweise. Auch hier begegnen uns, soweit Geometrisches in Frage kommt, nur die unbewiesenen Lösungen numerischer Berechnungsaufgaben, ähnlich wie im Papyrus Rhind. Aber die Aufgaben 10 und 14 des Moskauer Papyrus ragen, so schien es, durch ihre Schwierigkeit weit über alles sonst Bekannte hervor. Aufgabe 14 ist die richtige Berechnung des Rauminhalts eines Pyramidenstumpfes. Aufgabe 10 hielt der Herausgeber für die richtige Berechnung der Oberfläche einer Halbkugel. ARCHIMEDES rühmt sich in einem seiner Werke mit besonderer Genugtuung, als erster von allen Sterblichen die schwierige Kugeloberfläche berechnet zu haben, und es wäre wirklich eine Leistung, wenn die Ägypter sich an die Kugeloberfläche herangewagt hätten. Freilich müsste man auch dann sagen : Wenn zwei dasselbe tun, so ist es nicht dasselbe; denn zu den strengen Infinitesimalbetrachtungen, die Archimedes bei der Ausarbeitung der Lösung anwendet, wären

die Ägypter nimmermehr fähig gewesen, schon weil ihnen dazu das logische Rüstzeug fehlte. Vom Standpunkt der mathematischen Strenge müsste die Kugeloberflächenberechnung des ARCHIMEDES noch viel höher über einer etwa vorhandenen der Ägypter stehen, als seine Kreisberechnung über der ägyptischen stehen muss, wie auch immer die letztere ausgesehen haben mag. Aber textlich-sprachlich lässt sich aus der Aufgabe 10 nicht herausdeuten, dass die Berechnung einer Halbkugeloberfläche gemeint ist. PEETS Kritik (34) hat STRUVES dahingehende Deutungsversuche zunichte gemacht. Die mathematische Rechnung der Aufgabe 10 aber passt, wie ebenfalls PEET bemerkte, auch auf einfachere Gebilde, z.B. auf einen Halbkreis oder einen Halbzylindermantel. Wenn also auch die Aufgabe 10 noch nicht endgültig und eindeutig geklärt ist, so muss sie doch als Beweis für die höheren geometrischen Leistungen der Ägypter ausscheiden.

Es bleibt also die Pyramidenstumpfaufgabe. Es handelt sich hier um einen geraden regelmässigen vierseitigen Pyramidenstumpf mit der unteren Kante 4, der oberen Kante 2 und der Höhe 6. Die Rechnungen, nach denen der Ägypter den Rauminhalt 56 richtig findet, lassen sich in dem Ausdruck $(4^2 + 2.4 + 2^2).(\frac{6}{3})$ zusammenfassen, entsprechen also genau der Formel

$$v = (a^2 + ab + b^2).\tfrac{1}{3}h,$$

nach der auch wir den Rauminhalt dieses Körpers berechnen. Man hat es besonders bewundert, dass der ägyptische Geometer den „schwierigen" Faktor $a^2 + ab + b^2$ richtig gefunden habe, und einige haben hieraus geschlossen, dass ihm weitergehende mathematische Hilfsmittel zu Gebote standen, als man bisher glaubte (35), wobei man an Methoden dachte, die der griechischen und der modernen Mathematik nahe stehen. Andere (36) versuchten, die Pyramidenstumpfberechnung möglichst aus dem Wesen und mit den bisher bekannten Hilfsmitteln ägyptischer

(34) T. ERIC PEET, A Problem in Egyptian Geometry. *JEA* XVII, 1931, S. 100-106.

(35) So W. W. STRUVE in seiner Ausgabe des Moskauer Papyrus. *Qu. u. St.* A, I.

(36) GUNN und PEET, *JEA* XV, 1929, S. 167-185. K. VOGEL, *JEA* XVI, 1930, S. 242-249. W. R. THOMAS, *JEA*, XVII, 1931, S. 50-52. Meinen eigenen Lösungsversuch in der *Zeitschr. f. math. u. nat. Unterr.* 61, 1930, S. 145-158 möchte ich durch den im vorliegenden Aufsatz mitgeteilten ersetzen.

Mathematik zu erklären. Auch ich glaube, dass die Pyramidenstumpfberechnung auf dem Felde ägyptischer Geometrie, so wie ich dieses Feld im Vorstehenden zu kennzeichnen versucht habe, sehr wohl wachsen konnte. Da mich aber jene andern Erklärungsversuche nicht voll befriedigen, so will ich zeigen, wie ich mir die Sache zurechtgelegt habe, schon bevor ich in jene anderen Veröffentlichungen Einblick nahm. Ich muss dabei etwas weiter ausholen.

Zunächst ist nachdrücklich zu betonen, dass auch dieser Pyramidenstumpf ein schöner, regelmässig gebildeter Körper ist, der in der ästhetischen Phantasie des Ägypters lebte als Vervollkommnung der alten Mastaba, die nicht so ebenmässig gebildet war, weil sie eine rechteckige Grundfläche hatte. Zur Kennzeichnung des ihm geläufigen Körpers genügt dem Schreiber als Hieroglyphe das geradansichtig-vorstellige Bild des Körpers : ein stehendes gleichschenkliges Trapez .

Wie konnte nun der Ägypter überhaupt zu richtigen Berechnungsvorschriften für Flächen- und Rauminhalte gelangen?

Es trifft nicht zu, dass wir *gar nichts* von der hierbei eingeschlagenen Methode wissen, aber alles, was er uns über sie verrät, liegt in den drei Wörtchen beschlossen : *r rdí.t ífd.ś*, deren freie Übersetzung lautet : „um es rechteckig zu machen". Diese kommen einmal in Aufgabe 51 und einmal in Aufgabe 52 des Papyrus Rhind vor. Die jetzt von verschiedenen Seiten aner-

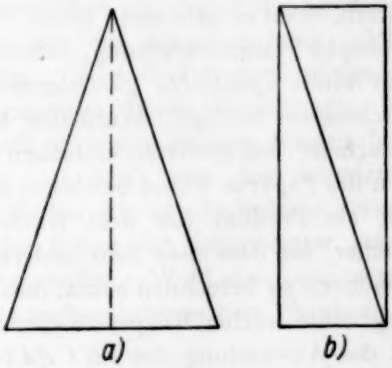


FIG. 2

kannte Deutung des Zusammenhangs, in dem die Worte gebraucht werden, ist diese : Um ein gleichschenkliges Dreieck

rechteckig zu machen, muss man seine durch die Höhe zur Basis erzeugten symmetrischen Hälften, d.i. zwei kongruente rechtwinklige Dreiecke (Fig. 2*a*) richtig aneinanderlegen, dann hat man das Rechteck (Fig. 2*b*). Entsprechend kann man ein gleichschenkliges Trapez auf verschiedene Weise, z.B. nach Fig. 3, „viereckig machen". Welch wichtige Etappe ist doch im geistigen Fortschritt erreicht, wenn die Menschen, wie wir es hier für die Ägypter annehmen, eine *geometrische Hilfslinie* ziehen!

Die Möglichkeit, eine Figur in eine endliche Anzahl von Teilfiguren zu zerlegen, und diese Teilfiguren zu einer anderen Figur zusammenzufügen, ist ein grundlegender Begriff der heutigen wissenschaftlichen Geometrie. Man sagt, das Dreieck Fig. 2*a* sei dem Rechteck Fig. 2*b* *zerlegungsgleich*, und ebenso das Trapez Fig. 3*a* dem Rechteck Fig. 3*b*.

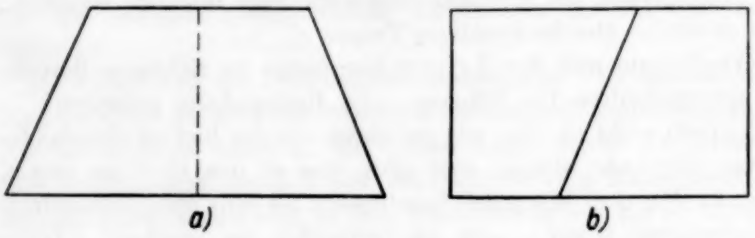


FIG. 3

Beim Versuch, den *Kreis* nach diesem Verfahren *ifd* zu machen, d.h., in ein Rechteck oder ein Quadrat zu verwandeln, mussten die Ägypter scheitern, denn es geht eben nicht. Aber sie fanden (wie?) eine vortreffliche Näherungslösung, indem sie den Kreisinhalt dem Inhalt eines Quadrates gleichsetzten, dessen Seite 8/9 vom Kreisdurchmesser beträgt. Wohl aber hat die Methode des „Rechteckigmachens" bei gewissen einfachen *Körpern* Erfolg. Mehrere Aufgaben des Papyrus Rhind beweisen, dass die Ägypter den Quaderinhalt als Produkt der drei Kanten berechneten. Nichts ist natürlicher, als dass man nun andere, nicht quaderförmige Körper dadurch zu berechnen sucht, dass man ihre Teile oder mehrere kongruente solcher Körper zu Quadern zusammenfügt. Das ist ja die Anwendung des *rdi.t ifd.ś*-Verfahrens auf den Raum, wobei daran erinnert sei, dass *ifd* such das räumliche „Rechteck", den Quader, bezeichnet. Diese Verwandlung in einen Quader gelingt z.B. leicht beim *Prisma*. Jedoch wissen

wir heute, dass eine beliebige *Pyramide* oder ein beliebiger *Pyramidenstumpf nicht* zerlegungsgleich einem Quader ist, d.h. wir können auf keine Weise eine solche Pyramide oder einen solchen Pyramidenstumpf durch ebene Schnitte in eine endliche Anzahl von Stücken zerlegen, die sich zu einem Quader zusammenfügen liessen. Den endgültigen Beweis für diese wichtige Tatsache, die die Griechen vielleicht ahnten, und von der man in der Neuzeit schon lange so gut wie überzeugt war, hat bekanntlich erst der zeitgenössische Mathematiker DEHN geliefert. Wohl aber giebt es spezielle Pyramiden und Pyramidenstümpfe, die einem Quader zerlegungsgleich sind.

Nichts hat den Ägyptern ferner gelegen, als die allgemeine Frage nach der Zerlegungsgleichheit von Pyramiden oder Pyramidenstümpfen mit Quadern aufzuwerfen. Wohl aber konnten und mussten sich ihnen einfache Einzeltatsachen aus diesem Kapitel der Geometrie aufdrängen, und zwar um so eher, wenn sie mit ihrer Freude an schöner Regelmässigkeit und mit der Betätigung dieser Freude in Kunst und Kunsthandwerk zusammenhingen.

Die Seitenwände vieler Särge des Alten und Mittleren Reiches (37) sind in einfacher Gehrung aneinandergefügt. Das sieht schöner aus, als der stumpfe Stoss, der den Anblick der Kante unsymmetrisch macht und die unschönen Schnittflächen der Bretter sehen lässt. Gelegentlich ist auch der Boden durch Gehrung angefügt (38). Fig. 4*a* stellt eine würfelförmige Kiste vor, bei der alles so ideal schön und regelmässig zusammengefügt ist, wie man es sich nur ausmalen kann. Alle Kanten haben nämlich Gehrung. Die 6 Bretter sind infolgedessen kongruente regelmässige vierseitige Pyramidenstümpfe. Kein ägyptischer Tischler hat wohl je einen solchen Kasten hergestellt. Schon aus technischen Gründen hätte das Schwierigkeiten geboten. Zum mindesten finden wir zwecks festerer Zusammenfügung bei den Särgen in den Ecken der Seitenbretter Laschen, die auf das Nachbarbrett übergreifen. Wohl aber ist eine solche Zusammenfügung der ästhetisch-geometrischen Phantasie angemessen, die

(37) Vgl. P. LACAU, Sarcophages antérieurs au Nouvel Empire, I. II. Catal. génér. d. ant. ég. d. Musée du Caire. Le Caire 1903-05. HEINRICH SCHÄFER, Priestergräber und andere Grabfunde vom Ende des Alten Reiches bis zur griechischen Zeit vom Totentempel des Ne-user-re, Leipzig 1908.

(38) Vgl. LACAU, I, Nr. 28005, S. 16.

alles so regelmässig und ringsum-gleich wie möglich macht. Derselbe Zug zur idealisierten Regelmässigkeit, die zugleich

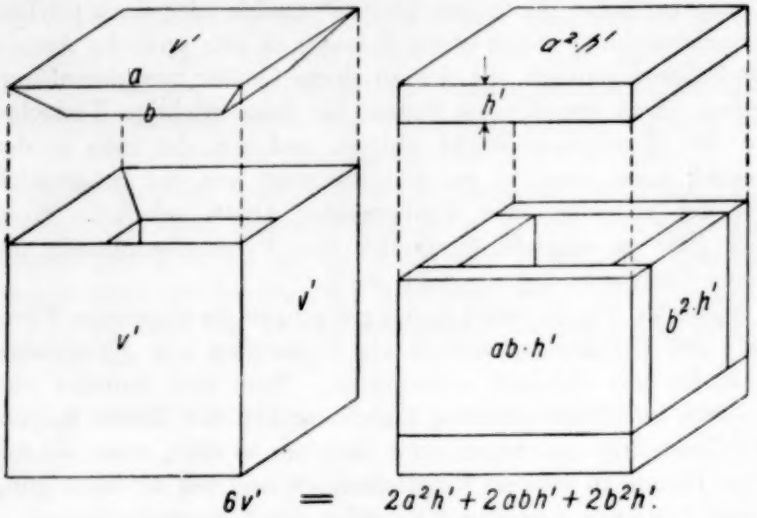


FIG. 4

geometrische Einfachheit ist, tritt uns entgegen, wenn in Rh 44 und Rh 45 ideal würfelförmige Kornspeicher berechnet werden. Oft hat dieser Zug sich auch verwirklicht, wie bei dem Übergang von der Mastaba, die eine rechteckige Grundfläche hatte, zur

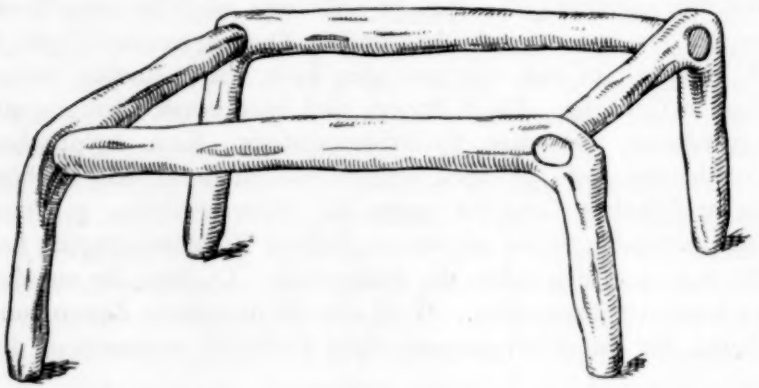


FIG. 5

reineren Form der quadratischen der Pyramide. Als Beispiel für eine geschlossene regelmässige Zusammenfügung gleichartiger Stücke sei auch ein Sessel des Heidelberger Ägyptischen Museums

genannt, der aus dem frühen Alten Reich stammt. (Fig. 5). Vier ineinander gesteckte gleichartige Krummhölzer bilden gleichzeitig den Sitzrahmen und die Füsse (39). Wir gehen wohl nicht fehl in der Annahme, dass der Hersteller an der Regelmässigkeit solcher geschlossenen Zusammenfügungen seine besondere Freude hatte, wie man sich z.B. auch an der ringsum gleichen Regelmässigkeit des uralten Hakenkreuzes ⸭ erfreut. Und die Freude des Handwerkers war zugleich die Freude des „Geometers". Übrigens haben auch für viele moderne Mathematiker *Schliessungsprobleme* ihren besonderen Reiz.

Wie in Fig. 2*b* zwei kongruente rechtwinklige Dreiecke zu einem Rechteck zusammengefügt sind, so in Fig. 4*a* 6 kongruente regelmässige Pyramidenstümpfe zu einem rechtwinkligen Körper, der zwar kein Quader ist (sondern ein Hohlwürfel), sich aber leicht aus Quadern zusammensetzen lässt. Will man letzteres so regelmässig und schön wie möglich ausführen, so tut man es nach Fig. 4*b*. Hier ist die Kiste in stumpfen Stoss so aus 3 Paar quaderförmigen Brettern zusammengesetzt, wie wir es z.B. bei unseren Zigarrenkisten beobachten können. Zu derselben Zusammensetzung gelangt man, wenn man sich die Struktur des Sarges LACAU II, S. 136, Nr. 28123 vereinfacht denkt.

Bezeichnen wir die äussere Würfelkante (d.i. die Grundkante jedes der 6 Pyramidenstümpfe von Fig. 4*a*) mit a, die lichte Weite der Kiste (d.i. die obere Kante des Stumpfes) mit b, und die Dicke der Bretter (d.i. die Höhe des Pyramidenstumpfes) mit h', und nennen wir den gesuchten Rauminhalt des Pyramidenstumpfes v', so ist das Holzvolumen der Kiste nach der Fig. 4*a* gleich 6 v', nach Fig. 4*b* aber gleich $2\,a^2h' + 2\,abh' + 2\,b^2h'$, also ist

$$6\,v' = 2\,a^2h' + 2\,abh' + 2\,b^2h',$$
$$v' = (a^2 + ab + b^2)\cdot\tfrac{1}{3}h' \qquad (1)$$

Gedankengänge dieser oder ähnlicher Art traue ich einem ägyptischen Geometer des Mittleren Reiches, ja vielleicht auch des

(39) Dieser niedrige Sessel H. 525 ist erwähnt in H. RANKE, Koptische Friedhöfe bei Karâra. Berlin und Leipzig 1926, S. 11. Die obige Figur 5 soll nur die Zusammenfügung der Knüppel erläutern. Das teilweise erhaltene Geflecht ist weggelassen. Dieselbe Zusammenfügung hat ein im Berliner Ägyptischen Museum (Nr. 14620) befindliches vorgeschichtliches Holzgestell, das als Lager für die Hockerleiche diente. Vgl. ALEXANDER SCHARFF, Die Altertümer der Vor- und Frühzeit Ägyptens II, S. 152 und Tafel 34. Berlin 1929.

Alten Reiches, zu. Alle Vorstellungen sind „übersehbar" und sind „urwüchsig", insofern als sie sich im Rahmen der schönen, regelmässigen Gebilde bewegen und in naher Beziehung zu Vorstellungen bleiben, die dem Kunsthandwerker geläufig sind. Dieser konnte sich sogar weit schwierigere räumliche Zusammenfügungen vorstellen. Die Buchstabenausdrücke muss man sich natürlich durch ihr ägyptisches Äquivalent ersetzt denken, nämlich Zahlenbeispiele, bei denen aber die Anwendbarkeit der Rechenvorschrift auf andere Zahlenbeispiele ohne weiteres vorausgesetzt wurde. Vielleicht konnte man ja auch die Rechenvorschriften in Worten aussprechen, indem man den Stücken ihre geometrischen Benennungen gab. Der „schwierige" Faktor $a^2 + ab + b^2$ ist, wie mir scheint, bei dieser durchsichtigen Ableitung natürlich und unmittelbar in seiner schönen Symmetrie erschienen (40). Die Anwendung des distributiven Gesetzes (das „Ausklammern" von $\frac{1}{3}h'$) bot dem Ägypter in einem solchen konkreten Falle keine Schwierigkeit. Noch weniger die Division durch 6, statt deren man übrigens auch sagen kann: Jedes Brett v' von Fig. 4*a* ist das arithmetische Mittel der Inhalte dreier verschiedener Bretter von Fig. 4*b*.

Nun haben wir freilich durch diesen Zerlegungs- oder richtiger Zusammensetzungsbeweis erst den Rauminhalt v' desjenigen speziellen regelmässigen vierseitigen Pyramidenstumpfes gefunden, bei dem die Kante der Deckfläche um die doppelte Höhe des Körpers kleiner ist als die Kante der Grundfläche, oder, was auf dasselbe hinausläuft, des Stumpfes, dessen Seitenflächen unter 45° gegen die Grundfläche geneigt sind. (Der Ägypter hätte vielleicht von einem Pyramidenstumpf vom *śkd* 1 gesprochen). Wenn wir die Möglichkeit unterstellen, dass der Ägypter diesen Stumpfinhalt durch den angegebenen oder einen ähnlichen Zerlegungsbeweis gefunden habe, so erhebt sich die Frage, wie er dazu kam, bei Pyramiden von beliebiger Seitenneigung nach derselben Formel zu rechnen. Durch *Zerlegung* konnte er die Richtigkeit dieser Formel etwa für das Beispiel von M 14, wo die Böschung der Seitenwand 6 : 1 ist, nimmermehr beweisen,

(40) Die von GUNN und PEET (*JEA* XV, 1929, S. 167-185) unter Mitwirkung von R. ENGELBACH ausgearbeitete Ableitung führt zunächst zu der Formel $v = h\,[ab + \frac{1}{3}\,(a - b)^2]$, die dann erst durch besondere Betrachtungen in die vom Ägypter benutzte Formel übergeführt werden muss.

da dies mathematisch unmöglich ist. Für solche allgemeineren Pyramidenstümpfe oder Pyramiden lässt sich die Inhaltsformel streng nur durch eine Infinitesimalbetrachtung beweisen, wie es ja auch die Griechen gemacht haben.

Es ist die Vermutung ausgesprochen worden, dass die Ägypter den Pyramideninhalt *durch Versuche* (experimentell) fanden. Ebenso wie man eine angenäherte Kreisquadratur durch Wägen oder Messen eines Zylinders aus Nilschlamm oder Wasser ausführen kann, so kann man auch durch ähnliche Versuche feststellen, dass der Inhalt einer Pyramide 1/3 des Inhalts des Prismas von gleicher Grundfläche und Höhe beträgt. Eine wissenschaftliche Aufgabe durch einen Versuch, d.h. eine ausdrückliche Frage an die Natur lösen zu wollen, halte ich für einen so modernen Gedanken, dass ich Bedenken trage, ihn in dieser Form für die Ägypter des Mittleren Reiches anzunehmen, solange hierfür keine Unterlagen beigebracht sind. Eher halte ich es schon für möglich, dass die metrologischen und technischen Erfahrungen mit den Inhalten von zylindrischen Kornmassen und den Gewichten von Steinzylindern im Laufe der Zeiten zu einer rohen Gewinnung des Kreisinhalts führen mussten.

Aber ich glaube, dass in vorkritischen Zeiten eher als durch gewissenhaftes Experimentieren richtige oder falsche Ergebnisse gewonnen wurden durch reine Vorstellungstätigkeit, die nicht immer strenges Denken war, sondern auch mit falschen Assoziationen, unerlaubten Analogien und Fehlschlüssen behaftet sein konnte, aber doch auch wieder vorauseilend manches richtige Ahnen enthielt. Man muss beobachten, wie heute noch Kinder und mathematisch Ungeübte die kühnsten Schlüsse auf Grund unzulässiger Analogien und Verallgemeinerungen ziehen.

Wir nehmen an, der Ägypter berechnete richtig die Inhalte von gleichschenkligen Dreiecken und gleichschenkligen Trapezen. Wir wissen ferner, dass er die Inhalte von Zylindern und Quadern richtig zu berechnen wusste. In allen diesen Fällen tritt als Faktor des Inhalts die *Höhe* auf, und man sah, dass z.B. die 6 mal so grosse Höhe bei gleich bleibender Grundfläche auch den Inhalt sechsmal so gross machte. In diesem Sinne war dem Rechner die Vorstellung der Proportionalität des Inhalts mit der Höhe lebendig. Auch bei dem speziellen Pyramidenstumpf mit der Böschung 1 tritt nach der Formel (1) die Höhe (oder ihr dritter

Teil) als Faktor auf. Was liegt näher, als bei *allen* derartigen Pyramidenstümpfen den Rauminhalt proportional der Höhe anzunehmen? Dann erhält man für den beliebigen regelmässigen vierseitigen Pyramidenstumpf von der Höhe h den Inhalt

$$v = (a^2 + ab + b^2) \cdot \tfrac{1}{3}h, \qquad (2)$$

und nach dieser Formel arbeitet der Verfasser des Moskauer Papyrus. Freilich haben wir sie hier durch eine auf Analogie gegründete unerlaubte Verallgemeinerung erhalten.

Aber dieser kühne Schluss braucht darum nicht blind und anschauungslos zu sein. Ich würde die Verallgemeinerung heute einem Anfänger in der Geometrie folgendermassen plausibel zu machen versuchen: Denke dir, eins der pyramidenstumpfförmigen Bretter von Fig. 4*a* sei entgegen der üblichen Weise so geschnitten, dass die Brettflächen senkrecht zu den geraden und parallelen Fasern des Holzes verlaufen, und nun stelle dir vor, unter Beibehaltung ihres Querschnitts verändern alle Fasern ihre Länge im Verhältnis $h : h'$. (Zahlenbeispiel!) Dann verändert sich der Rauminhalt jeder einzelnen Faser, und damit auch der Rauminhalt des ganzen Körpers im gleichen Verhältnis, und somit ergibt sich für den neuen Körper die Formel (2). Je feiner ich die Fasern annehme, umso mehr scheint die Richtigkeit dieses Verfahrens einzuleuchten. Es gehörte der Scharfsinn griechischer Mathematiker und Philosophen dazu, um die Mängel dieser Betrachtung aufzudecken und sie durch ein logisch strenges Verfahren zu ersetzen, das *Exhaustionsverfahren*, dessen modernes Äquivalent der Begriff des *Grenzwertes* ist. Aber die grössten Mathematiker, wie ARCHIMEDES, haben es nicht verschmäht, ihre Ergebnisse zuerst durch solche unstrengen Betrachtungen zu gewinnen, und noch heute arbeiten viele Mathematiker und alle Techniker so. Selbst die Formulierung einer solchen unstrengen Denkweise durch eine Indivisiblenmethode, wie sie DEMOKRIT (41) für seine Pyra-

(41) Nach dem zuverlässigen Zeugnis des ARCHIMEDES hat DEMOKRIT als erster den Satz ausgesprochen, dass Kegel und Pyramide ein Drittel des entsprechenden Zylinders und Prismas sind. Auch den Pyramidenstumpf haben die Demokriteer behandelt, denn wir wissen, dass sie ihn κόλουρος πυραμίς, „stutzschwänzige Pyramide", nannten. (Schol. Coisl. 166 in Arist. de coelo III, 8, 304 *b* 4, S. 514 *b* 5 Brandis. Zitiert nach S. LURIA in *Qu. u. St.* B, II, 1932, S. 144.)

Wenn auch der atomistische Beweisversuch des DEMOKRIT griechisch sein wird, so bleibt doch die Möglichkeit bestehen, dass die Rechenvorschriften, die Formeln, für die Inhalte aus Ägypten stammten.

Im Zusammenhang damit sei auf eine terminologische Merkwürdigkeit hin-

miden- und Kegelberechnung ausgebildet haben mag, wäre aber für einen ägyptischen Kopf etwas viel zu Wissenschaftliches. Vielleicht erschien es dem Ägypter ganz selbstverständlich, dass, wenn ein Flächenstück oder ein Körper beliebiger Form in einer Richtung gleichförmig gedehnt oder gestaucht wird, der Flächen- oder Rauminhalt im Verhältnis der Verlängerung oder Verkürzung zu- oder abnimmt. Vielleicht hat er dabei gar nicht das Bedürfnis nach einem Beweis für diese plausible Tatsache gefühlt, die auch gar nicht formuliert zu werden brauchte. Mit der Vorstellung einer solchen Dehnung können wir uns die Lösung der Aufgabe M 6 schön zurechtlegen, indem wir uns vorstellen, dass das Rechteck vom Seitenverhältnis 3 : 4 in der Richtung der kurzen Seite zu einem Quadrat gedehnt wird. Der Inhalt der gedehnten Fläche steht hier zum Inhalt der ungedehnten im Verhältnis 4 : 3 der neuen zur alten Höhe. Bei den Aufgaben M 7, M 17 und Kah. Pap. VIII, 31-42 ist es ähnlich. Möglicherweise ist es aber schon viel zu gewagt, dem Ägypter die Vorstellung einer solchen Dehnung zuzuschreiben. Jedenfalls denke ich hierbei nicht an die mathematische Vorstellung des kontinuierlichen Prozesses („fliessende Grössen"), sondern an die Vergleichung des ungedehnten Zustandes mit dem gedehnten.

Vergrössere oder verkleinere ich eine ebene Figur, so dehne oder stauche ich sie, wie wir heute sagen, in zwei aufeinander senkrechten Richtungen nach demselben Massstabe. Solche Vergrösserungen und Verkleinerungen waren dem Ägypter geläufig: Jede Grundrisszeichnung, jede Massstabveränderung eines Bildes ist nichts anderes. Dass hierbei Flächen im Quadrate des linearen Massstabes zu- oder abnehmen, überträgt der naive Verstand leicht vom Quadrat, wo es leicht beweisbar ist, auf beliebige andere Flächen. Indem die Ägypter *jeden* Kreisinhalt nach der Formal $\mathcal{J} = \frac{64}{81} d^2$ ausrechneten, nahmen sie ohne weiteres

---

gewiesen. Wie die Ägypter den Pyramidenstumpf nannten, d.h. wie sie die Hieroglyphe lasen, wissen wir nicht. Den Stumpf des gleichschenkligen Dreiecks aber, das gleichschenklige Trapez, nannten sie *ḥ3k.t*. Das Wort hat als Deutzeichen die Schwanzhieroglyphe, im Einklang damit, dass das Verbum *ḥ3k*. in den Pyramidentexten, 673 c, bedeutet: „den Schwanz abschneiden" (PEET, The Rhind Mathematical Papyrus, S. 95). Sollte die „stutzschwänzige Pyramide" der Demokriteer die Übersetzung eines ägyptischen Ausdrucks für den Pyramidenstumpf sein? Oder haben die Griechen unabhängig von den Ägyptern dasselbe anschauliche Bild getroffen?

als richtig an, dass Kreisinhalte sich wie die Quadrate der Durchmesser verhalten. Auch die geistvolle Deutung, die PEET (42) der Aufgabe M11 gegeben hat, beweist, dass ihnen diese Proportionalität geläufig war, vorausgesetzt, dass die in dieser Aufgabe vorkommenden Klötze zylindrisch gedacht sind, was doch wohl der praktischen Natur der Aufgabe angemessener ist als der quadratische Querschnitt. Auch hier haben anscheinend erst die Griechen das Bedürfnis eines Beweises verspürt, der wiederum ein Exhaustionsbeweis sein musste.

Ich nehme also an, dass dem Ägypter die Proportionalität des Pyramidenstumpfinhaltes mit der Höhe etwa ebenso plausibel oder selbstverständlich erschien, wie die Proportionalität des Kreisinhaltes mit dem Inhalt des umbeschriebenen Quadrates.

Zu dem Versuch, die Auffindung des Pyramidenstumpfinhalts im Anschluss an die Figuren 4*a* und 4*b* zu erklären, sei noch Folgendes gesagt. Ich behaupte nicht, dass der Ansatz gerade in dieser Form der einzig mögliche wäre, oder gar, dass die Berechnung des quadratischen Pyramidenstumpfes notwendig gerade aus der ägyptischen Tischlerei entsprang. Im Laufe der Jahrhunderte oder Jahrtausende mögen verschiedene Ansätze gemacht worden sein, die ihre Anregung auch in Stein- und Böschungsarbeiten finden konnten. Denkt man sich in Figur 4*a* die Innenwände verlängert, so hat man in jeder Ecke die Zerlegung eines Würfels in drei kongruente Pyramiden. Zu derselben Würfelzelregung gelangt man durch die Aufgabe : Drei Balken von quadratischem Querschnitt in einer Ecke vollkommen symmetrisch so zusammenzufügen, dass jeder der Balken auf jedem der beiden anderen senkrecht steht. Auch die Aufgabe, einer Böschung mit 45° Neigung einen konvexen oder konkaven Knick von 90° zu geben, führt auf diese Würfelzerlegung. Nimmt man aber die Erkenntnis dieser praktisch und ästhetisch so elementaren Würfelzerlegung an, so ist der Weg offen nicht nur zum Rauminhalt der quadratischen Pyramide (zunächst wieder von der Böschung 45°), die sich aus vier Teilpyramiden des Würfels bilden lässt, sondern auch zum Rauminhalt ihres Stumpfes nach der dem *Hero* geläufigen Formel

(42) *JEA* XVII, 1931, S. 158. Vgl. auch NEUGEBAUER, *Qu. u. St.* B. I, S. 434.

$$v = h.\left[\left(\frac{a+b}{2}\right)^2 + \frac{1}{3}\left(\frac{a-b}{2}\right)^2\right].$$

Nach der neuesten Forschung (43) arbeiteten höchstwahrscheinlich schon die Babylonier um 2000 v. Chr. mit dieser Formel. In Fig. 6*a* ist der Stumpf von der Böschung 45° zerlegt in den Quader

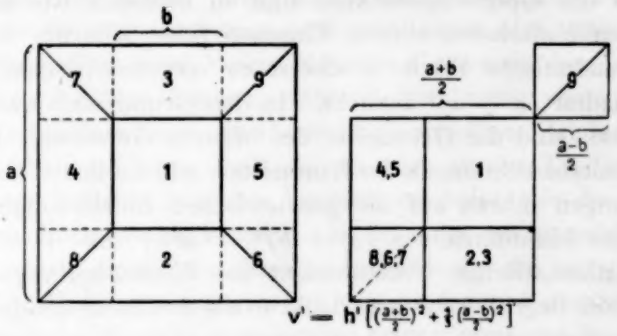


FIG. 6

1, die Prismen 2, 3, 4, 5, und die Pyramiden 6, 7, 8, 9. In Fig. 6*b* sind jeweilig zu einem Quader zusammengelegt die Prismen 2 und 3, die Prismen 4 und 5 und die Pyramiden 6, 7 und 8. So erhält man (Fig. 6*b*) einen Quader von der Grundfläche $\left(\frac{a+b}{2}\right)^2$ und die Pyramide 9 von der Grundfläche $\left(\frac{a-b}{2}\right)^2$.

Bei dem hohen Stand der babylonischen „Algebra" muss man freilich auch mit der Möglichkeit rechnen, dass die Babylonier im Stande waren, auf Grund ihnen geläufiger algebraischer Identitäten die Formel, die uns der Moskauer Papyrus erhalten hat, auf die bei ihnen vermutete Form zu bringen, oder umgekehrt.

Der Kern des Ansatzes nach den Figuren 4*a* und 4*b* ist eine Zerlegung eines *räumlichen Gnomons*. Denkt man sich durch den Mittelpunkt jedes dieser Hohlwürfel Ebenen parallel zu den Wänden gelegt, so zerfällt jeder von ihnen in acht zerlegte Raumgnomone. Dieser Zusammenhang mit der Idee des Gnomons

(43) Die Forschung stützt sich auf die freilich nicht ganz korrekt ausgeführte Rechnung in der Aufgabe CT IX, pl. 12, 41-49, die THUREAU-DANGIN behandelt hat in *Revue d'Assyriologie* XXIX, 1932, S. 87-88. Schon NEUGEBAUER (*Qu. u. St.* A I, S. 183) hatte bemerkt, dass hier ein Pyramidenstumpf nach genauer Formel berechnet ist.

erscheint mir wichtig. Die Idee des Gnomons ist eine der natürlichsten und fruchtbarsten der alten Mathematik, und man überblickt einen guten Teil der Geschichte der alten Mathematik, wenn man das Prinzip der Gnomonzerlegungen in seinen verschiedenen Ausstrahlungen verfolgt. Die praktischen Ausgangspunkte des Gnomongedankens sind so natürlich wie möglich. Auf den einfachsten ebenen Gnomon führt z.B. die Aufgabe, eine quadratische Fläche a^2 derart zu vergrössern, dass wieder ein Quadrat $(a + x)^2$ entsteht. In den Grundrissen einfachster Gebäude sind die Grundrisse der Mauern Gnomone. Die für die Inhaltsberechnungen des Baumeisters erforderlichen Gnomonzerlegungen führen auf die geometrischen Einkleidungen algebraischer Identitäten wie $(a + b)^2 = a^2 + 2ab + b^2$. Von der mathematischen Fruchtbarkeit des Gnomonprinzips erhält man einen Begriff, wenn man denkt an die Auflösung quadratischer Gleichungen durch Flächenanlegungen, an die angenäherte Quadratwurzelziehung, ferner an die Erzeugung „pythagoreischer Zahlen", an die Bildung der Dreieckszahlen, Quadratzahlen, Fünfeckszahlen, Pyramidalzahlen usw., an die Summierung der natürlichen Zahlen und der Quadratzahlen. Schliesslich steckt im Gnomon, als „Inkrement" einer Funktion y betrachtet (man denke etwa an $\Delta y = 2x.\ (\Delta x) + (\Delta x)^2$ und an die Integrationen des ARCHIMEDES) ja auch ein Keim zur Infinitesimalrechnung und zur Potenzreihe. Die Werke des ARCHIMEDES zeigen, wie stark noch dieser reife Mathematiker, in Ermangelung der Buchstabenrechnung, sein Denken an die anschauliche Form der Gnomonvorstellung klammerte, oder wenigstens diese Anklammerung für nötig hielt, um sich seinen Lesern verständlich zu machen.

Zum Schluss noch Bemerkungen allgemeinerer Art. Man hat wiederholt gemeint, dass die ägyptische Mathematik vielleicht auf einer von uns nicht geahnten wissenschaftlichen Höhe stand, die uns nur infolge der Spärlichkeit der zufällig gefundenen Dokumente verborgen geblieben sei. Sicher sind die uns erhaltenen mathematischen Schriften der Ägypter nicht die besten, die es gegeben haben mag, und die Mängel liegen nicht nur bei den beschränkten Kopisten, sondern auch bei den Verfassern. Dass aber weit über den Inhalt dieser Schriften hinausgehend im Mittleren Reich eine tiefe Wissenschaft vorhanden war, ist

höchst unwahrscheinlich. Das müsste sich in den vorhandenen Dokumenten auch dann verraten, wenn diese als Bücher und Schriften für den Elementarunterricht gedacht waren; ausserdem müsste das auch auf *den* Gebieten zutage treten, in denen sich die Mathematik auswirkt: in der Technik jeder Art, auch in der Technik der Kunst. Bei aller für solche allgemeinen Urteile gebotenen Vorsicht kann man doch wohl sagen, dass unmöglich die Mathematiker eines Volkes etwa die Geometrie der Zentralperspektive besitzen können, während gleichzeitig seine Maler und Flachbildner nur geradansichtig-vorstellige Zeichnungen kennen. Selbst für ein Volk, das so starr an der Tradition in der Kunst festhielt, wie die Ägypter, erscheint mir das nicht denkbar. Ebensowenig könnten z.B. bei einer gut ausgebildeten Trigonometrie die Sonnenuhren so primitiv wie die ägyptischen sein. Die Kulturäusserungen auf verschiedenen Gebieten sind eben in ihren Wurzeln miteinander verwachsen und bedingen sich bis zu einem gewissen Grade gegenseitig. Vor allem ist der Stand der sichtbaren Kulturäusserungen in Technik und Kunst in Harmonie zu dem Stand der reinen Wissenschaften: Mathematik und Philosophie.

Ich will nicht versuchen, die in der Überschrift dieses Aufsatzes aufgeworfene Frage durch eine *Definition* zu beantworten. Indessen hoffe ich, wichtige Gesichtspunkte für die Lösung dieser Frage hervorgehoben zu haben und glaube um so eher auf dem richtigen Wege zu sein, als meine Gedanken grossenteils an die für die Beurteilung der ägyptischen Kunst so fruchtbaren Ideen SCHÄFERS anknüpfen. Ägyptische Geometrie ist als vorgriechische Geometrie etwas anderes als unsere Geometrie, sie ist nicht ein geringeres elementares Quantum unserer Geometrie. Ein ungleichseitiges Dreieck, für unsere Elementargeometrie ein höchst wichtiges Gebilde als Inbegriff *aller* Dreiecke, als Dreieck *überhaupt*, ist, wenn meine Ausführungen das Richtige getroffen haben, für den ägyptischen Geometer ein belangloses Ding. Ein gerader quadratischer Pyramidenstumpf, für uns ein recht trivialer Sonderfall allgemeinerer Gebilde, mag für den ägyptischen Geometer ein Glanzobjekt seiner schönen „Wissenschaft" gewesen sein. Die Wert- und Lustbetonung ist anders verteilt. Gewisse methodische Begriffe und Forderungen, die uns selbstverständlich sind, fehlen dem Ägypter, und dafür wirken andere Triebkräfte.

In diesem Sinne stimme ich zu, wenn NEUGEBAUER (44) davor warnt, *unsere* geistigen Kategorien auf die Äusserungen ägyptischer Mathematik zu übertragen. Andererseits darf man aber auch nicht in übertriebenem SPENGLER*tum* sagen, dass diese ägyptische „Wissenschaft" etwas unserer Seele ganz fremdes und Heterogenes sei. Denn dann müsste sie uns ewig ein versiegeltes Buch bleiben. Dann hätten auch die Griechen nicht bei den Ägyptern in die Schule gehen können. Auch NEUGEBAUER sagt, dass es nur *eine* Logik gibt. Aber darüber hinaus wird man sagen können, dass auch jene „translogischen" Triebkräfte *unserer* nachfühlenden Seele nicht völlig unzugänglich sind. Denn wir finden solche Triebkräfte in den verschiedensten Formen und Ausbildungsstufen auch bei jetzt lebenden Menschen. Nicht nur die Kindesseele und die Seele sogenannter primitiver Völker deckt sie uns auf; auch bei reifen und „gebildeten" Zeitgenossen begegnen sie uns. Ganz abgesehen davon, dass auch heute noch die Zahl der Astrologen, Mystiker, Propheten und Ästheten nicht knapp ist, die die Betätigung ihrer translogischen Triebe in voller Überzeugung als „Wissenschaft" ausgeben, eine Wissenschaft, mit der die massvolle, bescheidene „Wissenschaft" der Ägypter oft den Vergleich gut aushält, — abgesehen davon, sage ich, gibt es auch heute noch viele, weder überschwängliche noch überspannte Menschen, welche sich auf Grund ihrer Naturanlage eine beachtenswerte primitive vorgriechische Wissenschaft für ihren Hausgebrauch zurechtmachen, die für den Betrachter oft reizvoll ist. Eine besonders dankbare Ausbeute bieten in dieser Hinsicht *die Dichter* (45). Wir sehen diese translogischen Kräfte auch in der griechischen und nachgriechischen Wissenschaft wirksam. Die Pythagoreer, bei denen auch die Zahlenmystik

---

(44) O. NEUGEBAUER, Die Grundlagen der ägyptischen Arithmetik. Bemerkungen zu einem Buch dieses Titels von Dr. K. VOGEL. (*Archiv f. Gesch. d. Math., d. Naturw. u. d. Techn.*, XIII, 1930, 92-99).

(45) Als Beispiel nenne ich die eigentümliche Vorstellung, die sich WILHELM RAABE an einer Stelle der „Chronik der Sperlingsgasse" über die Verteilung von Tag und Nacht auf die Erdkugel durch den Sonnenlauf oder die Erdumdrehung macht. Die künstlerisch schöne Schilderung enthält einen krassen Verstoss gegen die naturwissenschaftliche Kontinuität des funktionalen Vorgangs, wie wir solche Diskontinuitäten in der Vorstellungswelt des Kindes finden und auch nach unseren obigen Ausführungen bei den Ägyptern. (Siehe *Zeitschr. f. math. u. nat. Unterr.*, Lustige Ecke, 62, 1931, S. 432 und 63, 1932, S. 102.)

zu beachten ist, wurden schon erwähnt, ebenso Plato und Kepler. Diese alle beweisen uns, dass jene translogischen Kräfte, besonders wenn sie ästhetischer Natur sind, mit hoher Wissenschaftlichkeit in unserem Sinne verbunden sein können, wie wir das auch bei einzelnen Zeitgenossen beobachten. Ich erinnere an die Schönheitssucher unter den Kristallographen. Deswegen hüten wir uns, über die Bedeutung solcher translogischer Antriebe auch für unsere Wissenschaft voreilig den Stab zu brechen. Letzten Endes können uns jene Kräfte ja auch nur von Interesse und verständlich sein, weil sie, wenigstens im Keime, *in uns selbst* vorhanden sind. Denn wie der Tragödiendichter, wenn er einen Mörder oder einen Geizhals schildert, in sich selbst einen Keim jener verwerflichen Triebe haben muss, der natürlich durch andere, stärkere Triebe gehemmt ist, so müssen auch in uns, wenn anders wir Anteil an den Motiven vorgriechischer Wissenschaft nehmen, Keime dieser Triebe stecken. Insbesondere wird man Aufschlüsse erhoffen dürfen, wenn man sich in die vorwissenschaftlichen und wissenschaftlichen Gedanken und Vorlieben der eigenen Kindheit zurückversetzt. Denn wie für die Entwicklung des Leibes, so gilt auch für die des Geistes ein „biogenetisches Grundgesetz", das hier wie dort mit allen Vorbehalten und unter Berücksichtigung aller Unterschiede anzuwenden ist, die zwischen ontogenetischer und phylogenetischer Entwicklung bestehen (46). So kann umgekehrt die Beschäftigung mit der Geschichte ältester Kultur auch dem Pädagogen der Gegenwart Gewinn bringen. Bei der Einführung in die Bruchrechnung kann er sich der Schwierigkeiten bewusst werden, die der Begriff des allgemeinen Bruches der Menschheit bereitet hat.

---

(46) Wenn der Ägypter z.B. das gleichschenklige Dreieck liegend zeichnet, der Knabe der Jetztzeit aber stehend, so dürfte beim Ägypter schon ein aus langen „wissenschaftlichen" Erfahrungen entsprungener Brauch reifer Menschen vorliegen. Die liegend gezeichnete Figur bedeutet nach der von STRUVE erkannten Regel eine ebene Figur, die stehend gezeichnete einen Körper.

Dass man hier aber überhaupt von liegenden und stehenden Figuren sprechen kann, ist wieder ein Beweis für unsere These, wonach es sich nur um Figuren von besonderer Gestalt handelt, die Gegenständen der sinnlichen Körperwelt nahestehen. Wie wollte der Ägypter das „Viereck überhaupt", für das er nach STRUVES Meinung einen Fachausdruck besass, liegend zeichnen? Höchstens konnte er eine willkürlich als „Basis" angenommene Seite senkrecht zeichnen.

Wenn so mancher seiner Schüler, auch der höheren Stufen, immer wieder dazu neigt, statt eines beliebigen Dreiecks ein gleichschenkliges an die Tafel zu zeichnen, mag der Lehrer daran denken, dass sich hier eine ältere Schicht „wissenschaftlicher" Betätigung der Menschen Durchbruch sucht (47).

Berlin-Dahlem. P. LUCKEY.

(47) Den Herren Dr. Jos. E. HOFMANN in Nördlingen, Prof. H. RANKE in Heidelberg, Prof. H. SCHÄFER in Berlin danke ich für wertvolle kritische Bemerkungen und Winke, die sie mir bei der Abfassung dieser Arbeit gaben. Nach deren Abschluss sah ich noch ein: TH. E. PEET, Mathematics in Ancient Egypt. Reprinted from „*The Bulletin of the John Rylands Library*", Vol. 15, No. 2, July 1931, Manchester. „Der Verfasser glaubt, dass die Ägypter die richtige Formel, halbe Grundlinie mal Höhe, für das ungleichseitige Dreieck gefunden hatten, obwohl es ihm zweifelhaft ist, ob aus dem uns zur Verfügung stehenden Material ein streng logischer Beweis hierfür gegeben werden kann" (S. 24). Übrigens spricht PEET (S. 27-28) gegen STRUVE ähnlich wie ich oben unter (16) von der Möglichkeit, dass mit der „*Länge*" der Dreiecke in M7 und M17 die *Höhe* eines nicht rechtwinkligen Dreiecks gemeint ist.

Nach Abschluss der vorliegenden Arbeit erschienen folgende Veröffentlichungen: H. WASCHOW, *Archiv f. Orientforschung*, **8**, 215. K. VOGEL, *Arch. f. Orientf.*, **8**, 220. O. NEUGEBAUER, *Qu. u. St.*, **2**, 347.

Roger Bacon as Professor. A student's notes

M.S. 16089 of the Bibliothèque Nationale at Paris consists of a number of miscellaneous tracts. One of them contains what are most probably the lecture notes of a student at the University of Paris who attended the lectures of ROGER BACON on Mathematics. The manuscript is in a roughly formed thirteenth-century hand with a number of errors and omissions, due probably to carelessness or inability to get down all the teacher was dictating. It contains a note on measures, giving the length of digits, palms, feet, paces, and perches in barley corns, and the technical names of the sides of a right-angled triangle. A second note on arithmetical problems —*Cautele* or *Astucie Algorismi*—follows. They are problems such as to share 12 apples among 12 horsemen, footmen, and girls, so that each horseman has two, each girl a half, and each footman a quarter. How many of each class are there? The answers are given, but not the solutions. The two lectures here printed come next, and are followed by some calculations founded on the well-known treatise of SACROBOSCO on the Sphere.

The first lecture is headed *Reprobaciones Rogeri Baconis* from its explicit. The title may be taken as equal to 'criticisms' or 're-statements.' The subject matter is the accuracy of EUCLID's definitions. BACON valued the propositions of EUCLID as statements of fact, not as exercises in correct reasoning, but he insists on clear thinking in the definition of his terms and anticipates some modern criticisms of them. The first subject treated is continuous quantity, beginning with the definition of a line. A finite line demands a definition of a point. The next consideration is of a superficies and of extension. Then comes three-dimensional quantity—a body—, and lastly the divisibility of quantity. The next point considered is whether an angle is a species of quantity

or a figure, and a consideration of figures superficial or corporeal ; and lastly a note on ' locus,' a question arising in the *Physics* of ARISTOTLE.

The second lecture on the uses of mathematics is important as showing that in the *Opus Majus* BACON incorporated a quantity of previously written matter. Our student begins badly by mis-stating BACON's views as to the conflict between free will and the stars, and he uses some words which falsify his first reference to BACON's Greek grammar. The reference to the verse giving the difference between mathesis and mathésis occurs often in BACON. His *Metaphysics* was also in existence at this time. After magic the next subject treated of is Arithmetic; CASSIODORUS is quoted after BOETIUS, ISIDORE is also mentioned. Music is mentioned. Mathematics is necessary to the study of logic and music. Geometry is specially useful. The *Posterior Analytics* cannot be understood without it. A mention of HERMANNUS ALEMANNUS the translator is made. The next subject treated is verse and rhythm, quoting HORACE, and the lecture closes with a note on definitions.

A new lecture probably begins p.16 with a note on simultaneity and continuity. The difference between ' positions ' and dimensions in space is treated of at some length, and the lecture notes close with considerations on infinity, and on planes and curves, etc. The note taker has found himself unable to keep up with the lecturer and leaves half his sentences as ' etc.' probably trusting to his memory.

(Savage Club London.) ROBERT STEELE.

⟨REPROBACIONES ROGERI BACONIS⟩

Quantitas (1) vero secundum quid, aliud dicitur equale vel inequale. Hec descripicio sufficit pro quantitate in communi, nec alii rei habet in mente sequenti. Dicamus prius que pertinent ad quantitatem continuam, ideo quod in usu mathematici prior

(1) Hic sunt descripciones quantitatis et suarum specierum, et descripciones sunt EUCLIDE false assignate, quamvis sit hic proprium mathematice set communes sunt casus rithmi; in quibus debet probaciones mathematicas videre vulgus mathematicorum sciencia caret. *(in marg.)*

est continua quantitas. Diffinicio puncti non potest dari nisi prius describatur lineam continuam ut habeamus lineam finitam cujus termini sunt puncta. Est autem et linea quantitas continua permanens, habens posicionem unam sine latitudine et profunditate, que secundum veram consideracionem non est « longitudo sine latitudine, cujus extremitates sunt duo puncta », licet velit hoc EUCLIDES; quoniam linea non est longitudo, eo quod longitudo est posicio sive dimensio, et hec non sunt quantitates set denominant eas, ut linea posita in longum et dimensionata est longa, et ideo non posui linea de quantitatibus per inherenciam essencialem, set per denominacionem, et sunt proprietates quantitatum, nec deffiniunt propria vocabula sui generis et differencie quod in recto posset de ea parte secundum quod jam finita est. Continua linea curva non habet proprie longitudinem, set posicionem in girum, set cui esset ibi proprie longitudo esset rectitudo, set nihil rectitudinis (f. 92, b. 2) est in ea.

Item linea, unde finita est, habet extremitates et puncta, et quamvis omnis linea sit finita, nec possit esse nec intellegi infinita, ut dicit ARISTOTELES secundo *Methaphysica* et tertio *Physicorum*, linea tamen habet extremitates in eo quod est finita, non autem in quantum est linea, et ideo ad evidenciam distans debet apponi finitum, quando volumus dare linee puncta in extremitates.

Continuum bene concedit ARISTOTELES locis predictis nos possumus ymaginari in infinitum, et AVICENNA secundo *Methaphisice* vult quod quantitas non unde quantitas debet dici finita necessario, et ARISTOTELES hoc vult sexto *Physicorum ;* et philosophi ante eum usi sunt infinito et ponerunt ipsum esse in actu, ut mundum infinitum; et alii spacium vacuum esse extra celum infinitum, et ideo advocant planiciam distans linee per puncta debet apponi finitum ne ymaginacio nos perturbet. Et mathematicus accipit lineam quantum ipse vult infinitum, ut ARISTOTELES vult tercio *Physicorum*, et patet exquisitius mathematico. Set ante lineam finitam oportet punctum cognosci, quod in ejus definicio cadit.

Punctus vero non debet describere per hoc quod « est cujus pars non est », licet hoc dicat EUCLIDES, quoniam hoc contingit unitati et multis aliis, set est terminus linee finite, et e contrario. Curva linea est que ubique in girum equaliter declinat a rectitudine; linea curva convexa, que nata est circumscribi; concava est nata

circumscribere. Set tamen eadem linea est simul et semel per diversos respectus concava et convexa, scilicet, ut nata est continere vel contineri, quia linea non habet profundum quod facit concavitatem distare a convexitate; unde non sunt diverse species linee, set una est curva que diversas habet posiciones secundum diversas comparaciones et raciones.

Superficies est quantitas continua permanens, habens duas posiciones sine profundo, et ideo, sicut dictum est de linea, cum dicitur quod « superficies est que longitudinem et latitudinem tantum habet, cujus terminus sunt linee », licet hoc velit EUCLIDES, minus bene dicitur, quia superficies, unde finita, habet lineas in suis extremitatibus. Et nota : non omnes superficies habent longitudinem set solum plana; curva enim non habent longitudinem proprie loquendo, set habent posicionem dupliciter secundum latum et circuitum, et plana secundum latum et longum. Superficies igitur in universali est quantitas continua habens duas posiciones sine profundo. Plana enim superficies non est ab una linea ad aliam extensio in extremitates suas eas sustinens, licet hoc dicat Euclides, set est quantitas continua permanens secundum longitudinem et latitudinem.

Extensio enim est posicio que non ponitur faciliter de aliqua specie quantitatis, et plana superficies, unde finita est, habet extremitates circa distanciam nondum assignata : vel potest dici quod superficies plana est in qua possibile ⟨est⟩ protrahi ab omni puncto ad omnem punctum lineam rectam. Superficies concava est quantitas continua habens duas posiciones sine profundo, que potest pertransiri, vel cujus termini sunt linee.

Recta vero superficies non dicitur set plana, quia idem est rectum in lineis quod planum in superficiebus, set rectum sumitur quod habet longitudinem sine latitudine et profunditate, ut contingit linee, quoniam si rectum in omni sensu accipiatur, contingit superficiei, ut patebit. Set hoc non est in communi sermone, ut quando absolute loquimur de recto oportet enim arctari hoc nomen per aliquod determinale quando superficiei vel aliis conveniat.

Curva superficies que latitudinem sine profundo habet intelligitur (f. 93, a. 1) equaliter et ubique. Concava superficies est que latitudinem sine profundo in girum ubique habet aptam continere. Convexa superficies est que latitudinem in girum habet ubique

aptam contineri; quod cum non est profunditas insuper est, ideo una superficies secundum essenciam per diversos aspectus concavitatem et convexitatem habet, ut dictum est de linea.

Tertia species quantitatis continue est annectanda; facta enim est mensio de triplici dimensione et quadruplici posicione; set non est assignata aliter tercia, dimensio nec etiam posicio secundum profundum, et ideo ne suspendatur animus, dicendum est quod corpus est quantitas continua permanens que trinam habet posicionem:; nec debet addi « cujus termini sunt superficies » quia solum corpus, unde finitum est nature, est habere superficies. Et non difinitur corpus per dimensionem, quia AVICENNA secundo *Methaphysice* (2) hic reprobat, et licet sententia sua sit nimis dura, qua vult corpus unde corpus non habere dimensionum distincionem, quia in spera non sunt distincta, set possumus incipere ubique volumus et postponere dimensionem linee et deinde alias; tamen quia calumpnia de hoc non oportet quod corpus difinitur per distincionem dimensionum in actu, et loquor de quantitate sicut oportet suum sermonem, non solum intelligi de corporali substancia set de quantitate, set racio quam ponimus non habet hujus calumpniam, et ideo volui diffinire corpus per posicionem.

Corpus curvum ⟨est⟩ quod undique in girum habet superficies inclinatas equaliter. Corpus concavum est quod habet concavam superficiem interiorem. Corpus convexum est quod habet superficiem exteriorem convexam et interiorem concavam. Non debui diffinire planum in corporibus, licet aliquo sensu corpus dici posset planum, quia ante racionem et modos anguli non potest cognosci. Corpus planum habet enim angulum, nec est vere unum quoniam habet similes superficies ex quibus non fit superficies una.

Potui autem describere curvum ante planum in corporibus, quia in illa naturaliter primo est curvum, quoniam primum corpus nec habet superficiem unam curvam et convexam exterius et interius concavam. Unde aliter in corporibus est quam in lineis et superficiebus, quia rectum precedit (3) curvum in lineis, et planum precedit curvum in superficiebus, set in corporibus est e contrario.

(2) II, pars. 3, c. 1.
(3) Procedit MS.

Quomodo autem continuum est divisibile in infinitum, et hoc est in semper divisibilia. Linea non componitur ex punctis set ex lineis, nec superficies ex aliquibus indivisibilibus, nec corpus ex atthomis que dicuntur corpora indivisibilia, nec etiam superficies ex lineis, nec corpus ex superficiebus nec lineis, set linea omnis componitur ex lineis, et superficies ex superficiebus, et corpus ex corporibus.

De angulis dicemus aliquod notabiliter. Angulus secundum AVICENNAM tertio *Methaphysice* (4) sue est superficies vel corpus, nam quidam angulus est superficialis et hic angulus est superficies, quidam vero corporalis et hic est corpus. Angulus in communi est spacium contentum inter plures lineas in puncto aliquo concurrentes, quarum applicacio est indirecta. Angulus vero superficialis sive planus est spacium contentum inter duas lineas concurrentes in unum punctum, quarum applicacio in una superficie est indirecta, quia, cum directe componitur una linea alii non sit angulus, set recta linea una. Angulus vero corporalis sive solidum est contentus adminus in tres lineas concurrentes in punctum, quarum applicacio non est directa nec in una superficie set in corpore, ut patet in corporibus angularibus, quomodo (f. 93, a. 2) autem difinitur aliter angulus corporalis post ex *undecimo* EUCLIDIS. Et quoniam angulus superficialis est superficies, quia habet longum et latum, licet non ex omni parte clausum, et angulus corporalis est corpus quod habet longum, latum, et profundum, ideo impropie dicit EUCLIDES quod angulus est alterius contactus, nisi hic contactus erit indivisibilis secundum longum et latum et profundum, eo quod sit in puncto, et ideo difinivi angulum per quantitatem spacii divisibilis. Ideo ANARICUS in libro *Elementorum Geometricorum* (5) disputat utrum angulus sit species quantitatis nec ne, set non solvit.

Figura vero dicitur quasi curva vel curvis clauditur undique, ut quantitas contenta infra tres vel quatuor lineas, et secundum hoc angulus non est figura, licet sit superficies vel corpus. Si tamen figura dicatur quamvis non undique claudatur, tunc angulus est figura.

Angulus planus ⟨aliud⟩ est rectus, et est is qui ex casu linee recte super aliam nascens suo compari coequitur, et sic difinitur

(4) III, cap. 4.
(5) Ed. CURTZE (in EUCLID, *op. omn.* vi, TEUBNER).

sine linea perpendiculari, quod est melius quam considerando difiniciones sicut facit EUCLIDES, qui rectam lineam cum perpendiculari difinit. Angulus planus aliud est obliquus, et hujusmodi angulus est obtusus, et ille est major recto.

Post noticiam anguli noscatur linea connota, nam illa habet angulos rectos, ut vult Philosophus quinto *Methaphysice*, unde non est una nature, set composita ex duabus rectis, vel ex duabus curvis, vel ex curva et recta, etc.

Figurarum superficialium que undique clauduntur quedam clauditur linea una, que vero vocatur circulus, in cujus centro punctus est, et ad circumferenciam sunt equales : quidam vero duabus, ut semicirculus, qui ex medietate circumferencie et linea ejus extremibus que vocatur ' corda ' continetur. Dicitur etiam aliqua superficies linea curva in cujus tamen medio non contingit sumere punctum a quo esset linee ducte ad lineam continentem semper equales, et hic est quasi anularis figure (6). Si in piramide reducatur, erit tetragonus longus vel altera parte longior.

Corporalium figurarum quedam habent unam superficiem omnino, ut spera, quedam unam quidammodo set non simpliciter, ut corpus ovale et lenticulare, quedam duas superficies habent, ut porcio spere et semi-spera et piramus cujus rotunda basis graciliatur in rotundum, quedam ex tribus superficiebus, ut colupmna rotunda cujus sunt extremitates due circulares superficies inter quas corpus rotundum continetur. Quedam habent quatuor vel quinque, ut in piramidali laterata et colupmnali. Hoc sit dictum ad majorem dictorum evidenciam.

Post hoc oportet loqui mathematicum de quantitate continuo permanente extranea; ea sunt tempus et locus. Hec sunt extra rem localem et temporalem locus et locata, ut accidens in subjecto suo, cum comparatur ad rem extra. Non enim habet esse universaliter nisi per comparacionem ad aliquid extra substanciam locantis, et dicitur quantitas extranea, unde in astrologia et astronomia multum de hiis tractatur, quod cum aliis partibus quantitatis differitur.

Nota quod non possit superficies cognosci nisi cognoscitur locus, et e contrario; quoniam si superficies estimetur cognita per hoc quod est terminus corporis habentis duos dimensiones vel posi-

(6) Analis MS.

ciones sine profundo, cavillator (f. 93, b. 1) dicit quod hoc non videtur verum, quoniam locus potest esse alia species quantitatis, et est terminus corporis habens duas dimensiones vel posiciones tantum. Unde non videtur esse talis in quantum difinitur, quoniam alia species participat illam differenciam, tunc necesse est via cum diffinit locum cum specie,ut videat in qua communicant et in qua differunt. Et dicimus quod locus habet multa significata equivoce, et secundum suum esse potissimum est ubi superficies omnino concava ⟨est⟩, quantitatis habens comparacionem equalem ad profundum corporis concentri, et eundem respectum immutabilem ad terminos mundi; et secundum hoc locus non est alia species quam superficies, set habet tria accidentales; superficies concavitatis, et comparationem essentialem ad profundum concavitatis, et immutabilitatem in hoc ad terminos mundi. Et sicut partes mundi et celi omnes, preter ultimum et quatuor elementa, habent loca sua, quia superficies concava aque est locus terre naturalis, et precipue in illo loco ubi se conjungens etc., per ordinem; set celum ultimum non locatur secundum hanc concurrenciam, potissimam in aliis, licet reperiatur tamen plus in istis, etc. In concavitatem uniformem quod addit per comparacionem ad profundum corporis contenti. Hanc non exigit tamen superficies nisi esset illa profunditas, nec esset locacio, oportet enim latera et partes loci distare per profundum formalis contenti, licet dixerunt antiqui quod sic vacuum intercepit, quod in philosophia reprobatur. Expliciunt reprobaciones Rogeri Baconis.

⟨DE UTILITATE MATHEMATICE.⟩

Rogerus. Nec mirandum est quod in diversis scienciis multa tangant eadem, nam omnes sciencie sunt adinvicem connexe, et certum est eas in pluribus convenire, id est, mathematica. De hoc quedam est pars philosophie, et altissima illa inter mathematicas reputatur, hec est, illa que imponit necessitatem libero arbitrio, ut homo natus in tali constellacione sit castus necessario, et alius sit luxuriosus in alia constellacione natus; et sic de omnibus actibus docet quod a virtute stellarum necessitas sit in futura, ut nihil valeat racio aut consilium nec gracia Dei, nec temptacio

diaboli noceat homini in hac vita, et hec mathematica dicitur a « mantos » vel « mantia », quod est divinacio, ut in tractatu meo *Grece Gramatice* (7) explicavi. Set mathematica vera que cognoscat quantitatem et ejus speciem neutrum sciens nisi quod in hiis que de celo renovantur ut sunt eclipses et motus stellarum et hujusmodi potest omnino certificare, et potest ac debet racionem judicare; set de certificacione inferiorum rerum omnium, et de judicio infallibilli cum omnes et maxime communia humana, speculacio sua non presciit, veruntamen certificat celestia que sunt cause istorum inferiorum naturalium, et occasiones humanorum, per que potest homo peritus in rebus mundi et temperatus in judiciis quamplurima veraciter et laudabiliter judicare : et hec mathematica scribitur cum aspiratum, et dicitur a « mathesi », quod est disciplina, et habet mediam productam, ut in mea *Gramatica Greca* (8) exposui, licet totum vulgus Latinorum credat quod media sit correpta, secundum quod sint versus (9) sic :

Scire facit mathesis set divinare mathesis

Nota, qualis homo est justus sapiens talis in vitam est rectum, approbatur Aristoteles secundo *Methaphysice* (10), himis dificile aut impossibile ab errore recedere in consuetudinem.

Quamvis enim recitandum actoribus qui dederunt nobis sapienciam, tamen quia nichil est perfectum in humanis invencionibus, omnis enim res negligitur dum ejus laus et utilitas ignoretur, ostendi enim in *Methaphysica* mea per modos universales quod nulla scientia nec quod ⟨nulla⟩ ars potest suam ignoranciam in aliis percipere, nisi fuerit hic preclarissime instructus, nec res hujus mundi sciri possunt, nec utilitatem corporis nec rerum habere potest homo nisi hujus sciencie magnalibus sit imbutus, et circa primum sic procedo.

Incipiens (11) a BOETIO in prologo *Arismetice* (12) sue « mathematice quatuor partibus si careat inquisitor verum minime invenire potest », et iterum (13), « sine hac speculacione veritatis nulli

(7) Ed. NOLAN, Camb., 1902.
(8) NOLAN, p. 117.
(9) NOLAN, p. 118.
(10) Cf. ii, cap. 3, t. 14.
(11) Cf. *Op. Maj.*, I, p. 98.
(12) TEUBNER, ed. FRIEDLAN, I, 1.
(13) *Ibid.*, p. 9.

recte sapiendum est, » et ad huc dicit (14), « qui spernit has semitas sapiencie ei denuncio, » (f. 93, b. 2) et iterum ait (15), « constat quisquis hec pretermiserit omnem sciencie perdidisse doctrinam ». Quod etiam omnium antiquorum confirmat sentencia dicens (16) « inter omnes prisce auctoritatis (id est antique auctoritatis) viros qui Pictagora duce puriore mentis racione viguerunt, constare manifestum est; nec quemquam in philosophie disciplinis ad cumulum perfectionis evadere, nisi cui talis prudencie nobilitas quadrivie investigatur ».

Item (17) cum sint tres partes essentiales philosophie, ut dicit Philosophus in sexto *Methaphysice*, scilicet, Mathematica, Naturalis, et Methaphysica, non parum valet Mathematica ad reliquorum modorum duorum scienciarum comprehensionem, ut docet THOLEMUS primo capitulo *Almagesti*, unde BOETIUS in fine *Arismetice* (18) docet mathematicas ⟨medietates⟩ cum series in rebus (19) civilibus invenire.

Partes accidentales philosophie sunt Logica, Gramatica, Rethorica, et adhoc adicere multum valet Mathematica; quod patet per ea que dicit ALPHARABIUS in libro *De Scientiis*, et ARISTOTELES in libro suo *De Argumentis Poeticis*, et omnes actores Musice qui complete tractant eam hoc edocent evidenter.

Quare CASSIODORUS (20) in prefacione *De Artibus et De Disciplinis Scolarium* (21) sic ait, « mathematicam Latino sermone possumus doctrinalem appelare. » quo nomine licet omnia doctrinalia dicere valeamus, quemquem (22) docentur, hec tamen sibi commune vocabulum propter suam excellenciam proprio vendicavit. Et in tractatu Mathematice sic dicit (23) « discipline sunt que nunquam opinionibus decepte fallunt, et ideo tali nomine nuncupantur », et infert (24) « has dum frequenti meditacioni revolvimus, sensum acuunt, limumque ignorancie abstrahunt ».

(14) *Ibid.*, p. 9.
(15) *Ibid.*, p. 9.
(16) *Ibid.*, p. 7.
(17) *Op. maj.*, I, 99.
(18) *Loc. cit.*, II, 45.
(19) Omnibus MS.
(20) YSIDORUS MS.
(21) MIGNE, *Pat. Lat.*, 70.
(22) Queque MS.
(23) *Ibid.*, col. 1203.
(24) *Ibid.*, col. 1204.

Pro laude Geometrie dicit etiam; (25) « si fas est dicere, sancta divinitas quando creature sue diversas species formulasque dederit, quando cursus stellarum potencia veneranda distribuit, et statutis lineis facit currere que moventur, certaque sede que sunt fixa constituit. Quidquid bene disponitur aut completur, potest discipline hujus quantitatibus appellari ». Nam omnis creatura sensibilis sentenciam conficiat figura anime propria, sive pro motu, sive pro quiete, sive pro quacunque alia conditione et operacione, et secundum eleganciam ac cujuslibet rei privilegio consurgit.

Pro laude Arimetice dicit (26) : « Tolle numeros sive proporcionem a rebus et omnia ignorancia ceca complectitur, nec potest differe a brutis animalibus qui numeri ignorans quantitatem ». Et YSIDORUS (27) in tractatu *De Arismetica* in eandem scienciam eisdem verbis fere subjungit : « Adime seculo compotum, et cuncta ceca ignorancia complectitur, nec possunt differe a ceteris animalibus qui calculi non intelligit racionem ».

Pro laudibus et utilitatibus Astronomie dicit CASSIODORUS (28) : « Astronomia si casta et moderata mente perquirimus, sensus nostros, ut veteres dicunt, magna claritate perfundit. Quale enim est ad celos animo subire, et totam illam machinam supernam indagabili ratione discurrere, et inspectiva mentis subtilitate ex aliqua parte colligere, quod tante magnitudinis archana velaverunt », Et subjungit, « ex quibus, ut mihi videtur, climata noscere, horarum spacia comprehendere, lune cursum pro inquisitione pascali, ne simplices aliqua confusione turbentur, qua ratione fiant, advertere non videtur absurdum ». « Est alia quoque de talibus non despicienda commoditas, si oportunitatem navigacionis, si tempus arancium, si estatis caniculam, si autumpni suspectos imbres inde discamus ».

Musice laudes inducit in via et studio Theologie, et in multis et quatinus sciencia ipsius ejus utilitates infert sermonem ejus ultimum, et enim, ut breviter cunota complectar, quidquid in supernis sit tractatum (f. 94, 1. 1) rebus geritur ad hanc disciplinam non refertur exepta gratissima, ergo nimis utilisque cognicio que

(25) *Ibid.*, col. 1213. Cf. *Op. maj.*, I, 180.
(26) *Ibid.*, col. 1208.
(27) ISIDORE, *Etym.*, iii, 4, 4.
(28) MIGNE, *P. L.*, 70, col. 1218.

sensum nostrum ad superna erigit, et aures pro modulacione permulcet. Multas autem auctoritates sanctorum obmitto de laudibus harum scienciarum.

Item, laudes Mathematice et utilitates oriuntur ex consideracione eorum que in singulis scienciis de mathematicis requiruntur, set sciencialia (29) et logica precedunt mathematicam in ordine, et non debeo probare aliquid nisi eis qui possunt intelligere racione probandi, ideo, his que estimant scire Logicam et Grammaticam sine scire naturalia et mathematicalia et alia, sunt non. Volo probare quod non potest homo docere Logicam nisi sit optime in mathematicis imbutus, et tacebo hic de aliis, nisi nihi obiciatur quod utor eis in docendo que non possunt sciri ante Mathematicam; quoniam mathematicus docens potest descendere ad logicalia et grammaticalia, quia in eis instructus est. Si igitur vis Gramaticam veraciter docere, tunc certum est quod illa se habet ad Musicam sicut Carpentatoria ad Geometriam, et sicut quecunque Mathematica ad partem philosophie cujus est dare causas et racionem de hiis que in natura eis subjecta sunt; causarum noticia particularis, nam ad solam Musicam pertinet dare causas vocum humanarum secundum omnes suas proprietates abstractas. Similiter, quedam musica protracta, quedam metrica, quedam rithmica, etc. Unde grammatici istius temporis qui nesciunt causas et raciones metrorum et rithmorum et accentium et colarum et causa et peryodos, componunt ritmos et metros a casu. Sic Logica, quantum ad principium medium et finem, sine Mathematica sciri ⟨non⟩ potest, quia quantitas doceri est; liber *Predicamentorum* doceri non potest sine Mathematica neque cognosci, et omnia que ei sunt annexa, ut situs, ubi, habitus, quando.

Nam posicio, que vocatur situs, est ordinata respectu loci alicujus, ut sedere, jacere, stare, et AVERROES (30) quinto *Methaphysice* predicamentum posicionis sive 'ubi' non utitur sine comparacione, dico, ad ubi; et major pars qualitatis continet proprietates et passiones quantitatum habitus quorum Geometria et Arismetica constituitur; et omnes sunt in quarto predicamento quantitatis, et hec sunt proprietates linearum et angulorum, superficierum, et corporum et omnium figurarum et numerorum,

(29) Sola, MS.
(30) In ARIST. *Met.* V, comm. 24.

quibus ARISTOTELES qualitates in quantitatibus, ut rectum, curvum, figura, efiguratio, et omnia que catagorie quantitatis in mathematicis assignaverunt.

Amplius igitur de Geometria est consideracione in puncto universaliter, et proprietas quantitatis relate, ut super proporciones proporcionatas et medietates Geometrice et Arismetice et Musice; et super species majoris inequalitatis et minoris, et cetere proprietates relate quantitati, que in mathematicis pertractantur.

Amplius, nec accio nec passio possunt sine lineis angulis et variis figuracionibus cognoscere, quia ARISTOTELES dicit, quinto *Physicorum*, quod natura operatur brevissimam que potest; linea recta est brevissima, et ideo secundum eam est accio nature. Et hoc vult Philosophus secundo *Celi et Mundi;* dicit quod grave cadit deorsum ad angulos rectos, et sic est determinatum operacione naturali, nam illa linea prior est que cadit ad angulos rectos quam ea ⟨que⟩ cadit ad angulos obliquos.

Amplius; a quolibet puncto agentis superficiei venerit virtus ad quodlibet pacientis punctum. Hec autem est figura piramidalis, cujus basis est in agente et conus in paciente, et sub illo cono virtus a quolibet puncto agentis unitur forcius ad paciens sic.

De predicamentis accidencium nihil cognoscimus sine ea, nec substancia cognoscitur, quia spirituales substancie non cognoscimus per philosophiam nisi per corporalia, et maxime per corpora celestia, ut Philosophus dicit undecimo *Methaphysice*, Nec inferiores cognoscimus sine superioribus, quia cause inferiorum sunt ille, et celestia non cogniscimus nisi per quantitatem, sicut patet de Astronomia (f. 94, a. 2) libris; demonstracionum medium est eorum Logice influens, in ceteras partes non scientur nisi optimus sit mathematicus, nec loquor quantum ad exempla mathematica, quia he sunt signa omnibus scienciis, etc. Nam exemplum ARISTOTELIS (31), de addicione gnomonis ad quadratum minus ut fiat majus, non potest sciri ante vicesimam proposicionem *sexti libri;* et quod dyameter est assimiter non potest sciri ante ultimam partem septime proposicionis *decimi libri;* nec quadratura circuli scietur ante secundam proposicionem *duodecimi libri;* et sic de aliis exemplis tam in numeris quam in figuris, et astronomicis et musicis, que in omnibus scientiis inducuntur.

(31) Cf. *Op. Maj.*, I, p. 103.

Dico ergo quod liber *Posteriorum* sine Arismetica et Geometria non scietur. Communia, que sunt diffiniciones et proposiciones et peticiones et supposiciones simpliciter, et aliter et maxime et dignitates, sciri non possunt in libro *Posteriorum*, eo quod ibi ratio ponitur; ideo oportet quod doctor sciat addiscere de de scienciis ubi explicantur per ordinem omnia illa; hec autem est in *Geometria* et *Arismetica* JORDANIS, etc.

Nec potest purus logicus scire quid est quod per nomen dicitur in hiis, quia in illo libro non datur racio nominum, set confunditur et ocultatur, precipue propter malam translacionem que multum impedit, nec ARISTOTELEM expressit. Unde sicut liber PORPHYRII ad *Predicamenta*, sic Geometria ad librum *Posteriorum* preambulat, aut quod sciat introducere lector, quia sola mathematica vere est demonstrativa secundum demonstracionem positivam, de qua est liber *Posteriorum*. Et ubique fit demonstracio hoc est per beneficium mathematice, sicut in Perspectiva et in aliis scientiis.

Item, Philosophus ostendit quod potest discernere in supposicionem, et ponit diferenciam inter ea. Atque supposicio debet esse vel non-esse, et est proposicio affirmativa vel negativa, difinicio tamen non sic, nec est hoc quantum ad modum loquendi in mathematicis nec in principiis suis, set oportet demonstrare quod difinicio mathematica sit proposicio affirmativa vel negativa, ut patet in omnibus mathematicis; et oportet quod difinicio sit supposicio, ut ostendi in *Methaphysica*, et quod non sit probacio. Demonstrata non videntur, scilicet, supposicio et peticio et conceptum, quorum nomina tamen nec exponuntur in dicto libro, nec realis racio assignatur. Qui igitur *Posteriora* docet, nisi sciat naturam demonstracionis philosophiam non exponit, quem sciet quod diffinitum aliquando est demonstrabilis commune et aliquando physicum, et ubi logicus purus habebit de hoc exempla nisi in mathematicis.

Item, multa non potest purus logicus, ut deberet in libro *Posteriorum* demonstrare, ut exigit racio bene. Virtus libri *Posteriorum* ab eo non percipitur, nisi qui fuerit instructus in mathematicis.

Item, cognitio ejus valet ad librum *De Argumento Poetico* quam fecit ARISTOTELES in Greco, sicut ostendit HERMANNUS ALEMANNUS translator (32), qui se excusat quod textum ARISTOTELIS quem

(32) Cf. *Op. Maj.*, I, p. 101.

fecit de arte poetico non potuit transferre, quia ingnoravit Greca metra quibus utitur ARISTOTELES in hoc libro. Tamen iste liber translatus est in Arabicum de Greco, licet in Latinum non sit conversus textus ARISTOTELIS, set ALPHARABIUS exponit nobis intencionem illius argumenti, et AVICENNA et ALGAZEL et OVEROIS (33), unde oportet ponere geometrice argumentum poeticum. Et quod Logica, de hoc constat librum qui ad solam Logicam pertinet autorum composicio et doctrina; et hoc monstravi in *Methaphysica* quod hoc genus argumenti est necessarium in Logica, et quod argumentem hoc est utilius omni argumento, conferatur in anime salutem, et confert circa virtutes, [circa vicia,] circa felicitatem, circa vicia, ut deleantur. Et illud argumentum est proprium, ut in textu *Moralis Philosophie*, et in usu ejus, et similiter in theologicis proprietatibus et doctrinis, et plus potest tale argumentum movere sine omni comparacione quam demonstracio quantumcunque potissima habeatur.

Et exposui de eo tractatum in logicalibus, et ostendi quod exponit ALPHARABIUS (34). Hoc argumentum utilissimum est in sermonibus pulchris (f. 94, b. 1) et in fine decoris, ut rapiatur subito animus in amorem virtutis et felicitatis, et in odium vicii et pene perpetue que sibi respondet; et ideo sermones poetici qui sunt completi in pulcritudine et efficacia movendi debent exornari cum venustate (35) loquendi prosaice, et astricti omni lege metri et rithmi, sicut scriptura sacra in lingua primitiva, que Hebraica est. Et BOETIUS in *Consolatione Philosophie* et ALANUS *De Consolatione Nature* et poetria ORACII in litera *De Officio Divino*, ut decore et suadente sermonis animus fortiter et subito moveatur. Nam

> aut prodesse et iterum volont aut delectare poete (36)
> Omne tulit punctum qui miscuit utile dulci (37)

quive deliciis arrident seriam mixtam jocis. Hoc autem perfecte docetur, scilicet, composicio hujus argumenti per omnem sermonis prosaici [sermonis] eloquenciam, et per omnia metrorum et rithmorum genera, set cuncta hec pertinent ad Musicam, et

(33) AVERROES.
(34) Cf. *Op. Maj.*, I, 100.
(35) Vetustate MS.
(36) *Ars, Poet.*, 333.
(37) *Ibid.* 343.

melius liquebit infra, unde libri loyce sic non possunt doceri sine beneficio Musice, nec intellegi; cujus est causas et rationes per se et metrorum et rithmorum dare, et adhuc facit quod translator Latinus non transtulit nobis librum quia non intellexit, ut dicit in volumine *Metheororum* (38), raciones de quibus ARISTOTELES loquitur habundanter; non ergo poterit transferri nisi per hominem qui metrorum et rithmorum causas moverit. Manifestum ⟨est⟩ quod nec potest hic liber intelligi ut videtur causis istis nec ab aliquo doceri; et hic liber est finalis Logice, quia ultimum genus argumenti perdocet, quod, scilicet, ad ultimum bonum hominis, scilicet, virtutis et felicitatis, excitat vehementer.

Alia enim argumenta inducta movent ad opinionem, ut Dyalecticum, vel ad scienciam ut Demonstrat⟨iv⟩um, vel ad credendum ut Rethoricum, set illud ad faciendum inducit, et ordinat ad bonum hominis completum. Et sunt quatuor argumenta, de quibus est Logica ut libri ARISTOTELIS docent, finis meliorum eis que sunt ad finem, et imponit necessitatem illis, ut ARISTOTELES docet; et ideo hoc argumentum est dignius aliis, et qui illud nescit non poterit aliquid scire ut oportet, quia utilitas eorum que sunt ad finem non est nisi respectu ejus finis. Logice non potest sine Mathematica scire, sicut nec medium nec principium.

Nota, difinicio sumitur large modo pro descripcione in mathematicis quandoque, quia mathematicus non curat observare proprietatum difiniciones. Hoc magis ad methaphysicum pertineat, vocabulorum differenciarum specificarum que differencie conveniunt in pluribus non sunt imposita, et ideo circumloquimur per descripciones, ut possumus et debemus, et diffiniciones hujusmodi ponende sunt alibi, quoniam oportet presciri quid est quod dicitur per naturalis, nihil enim aliter potest sciri nisi loquimur hic de difinicione nominis que dicitur nominis interpretacio, ut ' homo ' dicitur ' factus ab humo,' set magis de difinicione reali cum qua tamen satis difinicio nominis innotescat.

***

Dico igitur quod ' simul ' vel est secundum locum vel secundum duracionem, et loquimur de sola temporis duracione. Simul autem loco secundum indivisionem ejusdem loci vel numeris, ut substancia et accidens, et corpus et anima que sunt in eodem

(38) Vide *Op. Maj.* I, p. 101 note.

loco, vel secundum concomitanciam duorum locorum sine medio, ut duo corpora inter que nullum est medium, habent due loca simul.

Simul duracione similiter, vel sunt aliqua per indivisionem ejùsdem mensure durancium, ut duo motus qui sunt in eadem hora temporis, duracione simul vel concomitanciam duarum mensurarum duracioni ejusdem speciei sibi invicem succedencium, quarum neutra sit pars alterius vel ambe ejusdem, ut due distincte hore sibi invicem succedentes, ut hora prima et secunda, et si in hora prima esset aliquis motus unus et non alius, at ille (f. 94, b. 2) cessaret omnino et statim alius motus renovaretur in hora secunda, etc., illi duo motus essent simul duracione per concomitanciam duarum mensurarum durandi ejusdem speciei sine medio, qui nec essent penitus ejusdem duracionis nec altera pars alterius. Et si motus celi et omnes motus simul concomitantes starent, et statim alius motus nasceretur sine tempore medio, quod bene est possibile, ille motus et motus celi precedens essent simul tempore secundum dictam signacionem consimilis dubitacionis.

Tertio modo est simul duracione per concomitanciam duarum divisionum diversarum secundum speciem, ut cum dicatur deus et motus celi simul sunt. Hoc non est quia esse eorum mensuretur eadem duracione, quia esse dei mensuretur eternitate, et motus ⟨celi⟩ tempore, que omnino sunt diverse mensure.

Terminus est cujus est finis vel principium, et hoc est ultimum etc. Vel extremitates contigua sunt, quorum termini vel ultima sunt simul per concomitanciam duorum locorum vel duarum duracionum sine medio, vel duo loca et dua locata sine medio juxta se posita; et duo tempora distincta, ut dictum est, consimili minute successive, quorum unum non est pars alterius, nec ambo partes ejusdem vel earundem, et motus mensurati illis temporibus sunt contigui. Concomitancia sunt quorum ultima sunt unum, ut due verge unius continuantur ad unum terminum, quando, vere scinduntur adinvicem habebunt terminos diversos.

Item, continuum est cujus termini copulantur ad terminum communem.

Item, aliquid [non] continuum est divisibile in semper divisibilia ut nunquam venitur ad indivisibile, et cum virga dividatur vel corda vel aliquid tale potest dividi in duas medietates, et quelibet

medietas in duo media, et sic semper fiat divisio in divisibilia, et nunquam pervenitur ad indivisibilia.

Discretum vero est cujus partes non copulantur ad communem terminum, ut partes tocius aggregati, ut in cumulo lapidum vel cinerum vel pulveris. Permanens est, quod simul duracione vel indivisionem ejusdem dimensionis secundum numerum totaliter esse potest, ut esse dei est permanens quod est in eternitate, et eternitas est tota simul. Successivum est, cujus parte posteriore adveniente non manet precedens, ut partes motus et temporis. Dimensio est deducio rei secundum longum, latum, vel profundum, et ideo patet dimensiones primo proprie dicta, vel est in quarto categorie quantitatis, et sic est ordinacio partium alicujus in suo toto, vel subjecto, non distincta hic et ibi secundum longum latum et profundum, vel certius, ut contingit figure in lapide, hic unam partem hic ibi aliam secundum dictas vias.

Quoniam igitur tres sunt dimensiones tantum, secundum ARISTOTELEM et omnes, et quatuor posiciones, non omnis posicio est dimensio, omnis tamen dimensio est posicio, quia ad posicionem requiritur permanencia, ut dicit ARISTOTELES in *Predicamentis*, et ideo in successivis est dimensio non tamen posicio. Motus enim habet dimensionem secundum dimensiones speciei, et tempus secundum omnem rem motus, set quia non permanent nec huc erit stabile set in continuo fluxu et mutacione, ideo non habent posicionem; non enim contingit significari ubi una pars poneretur nec alia.

Posicio que est in predicamento situs parcium respectu corporalis ordinacio in toto vel in subjecto respectu loci, ut jacere vel subter et aspectum et leve et hujusmodi. Et aliis modis non est posicio proprie, licet per equivocacionis motum habent suum prout ordine parcium in toto, et precipue respectus quantitatis, ut dicatur ordinacio parcium hujusmodi et parcium motus et temporis et omnium, set hic non est in usu philosophie.

Infinitum est quod pertransiri non potest, et, secundum ARISTOTELEM in tercio *Physicorum*, infinitum est cujus partem accipientibus semper remansit aliquid attendendum, sicut si virga extenditur in infinitum continget (f. 95, a. 1) semper accisio ad ultima, id quod acceptum est, et nunquam posset pertransiri, et sic dicitur quod continuum est divisibile in infinitum.

Infinita sunt non tot quot non plura, sicut ARISTOTELES secunda

De Generatione, et que pertransiri non possunt. Finitum est quod pertransiri potest, finita autem que comprehendi possunt.

Rectum est multis modis partibus equare, nec potest diffiniri in aliquo figurato qui omnibus modis quanto vero communis accipi potest in uno figurato. Estimo quod sic describi potest; rectum est quod totaliter caret obliquitate, et sic contingit non solum lineis set quibusdam angulis, racio enim facile esset dare. Aliquid commune lineis et angulo recto et hujusmodi ubi caret eam omnis obliquitatis. Multa enim sunt quorum partes diffinitas non habent nomina propria, et ideo describuntur per paronomasiam, et he sunt res nobiliores propter quorum vocem differencie et propria earum nomina non habent, ut per paronomasiam unum relativorum per alterum difinitur.

Planum uno modo idem est quod superficies, sicut ARISTOTELES et Greci ceteri utuntur pluries, set nunc interistas distincciones ponunt communium. Dico quod planum habens longum et latum circa omnes partes extenditur in rectum.

Curvum est quod a plano et recto totaliter declinat. Concavum est circumscribere, convexum quod natum est circumscribi et non circumscribere. Jam sunt multum necesse etc.

Leonhard Euler's Elastic Curves

(De Curvis Elasticis, Additamentum I to his Methodus Inveniendi Lineas Curvas Maximi Minimive Proprietate Gaudentes, Lausanne and Geneva, 1744).

Translated and Annotated
by
W. A. OLDFATHER, C. A. ELLIS, and D. M. BROWN

PREFACE

In the fall of 1920 Mr. CHARLES A. ELLIS, at that time Professor of Structural Engineering in the University of Illinois, called my attention to the famous appendix on elastic curves by LEONHARD EULER, which he felt might well be made available in an English translation to those students of structural engineering who were interested in the classical treatises which constitute landmarks in the history of this ever increasingly important branch of scientific and technical achievement. He secured photostats of that copy of the original publication which was owned by the New York Public Library, and together we spent many delightful evenings working over the translation, and correcting the occasional errors of printing and calculation which such a first edition inevitably contained. We also examined and translated a considerable number of the notes in Dr. H. LINSENBARTH's admirable translation and commentary (Leipzig, 1910). The Ms. was practically completed when Mr. ELLIS left the University in order to enter active business in Chicago. For some time the various drafts and annotations lay in my files, until early in 1932, when I was fortunate enough to secure the

METHODUS
INVENIENDI
LINEAS CURVAS
Maximi Minimive proprietate gaudentes,
SIVE
SOLUTIO
PROBLEMATIS ISOPERIMETRICI
LATISSIMO SENSU ACCEPTI

AUCTORE
LEONHARDO EULERO,
Professore Regio, & Academiæ Imperialis Scientiarum PETROPOLITANÆ *Socio.*

LAUSANNÆ & GENEVÆ,
Apud MARCUM-MICHAELEM BOUSQUET & Socios.

MDCCXLIV.

Tabula III. *Additamentum.*

Fig.1.

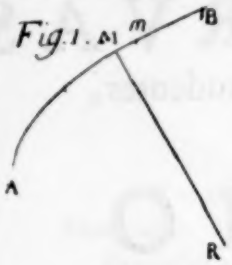


Fig.2.

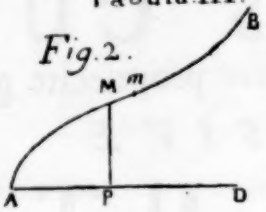


Fig.3.

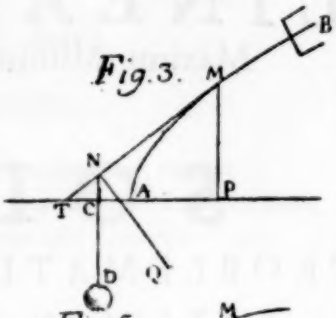


Fig.4.

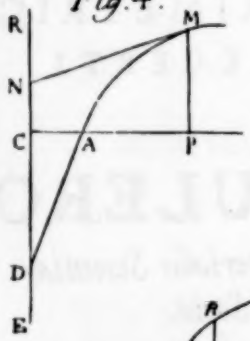


Fig.5.

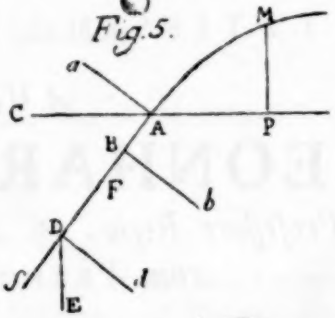


Fig.6.

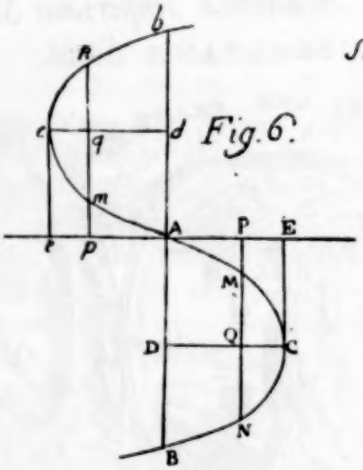
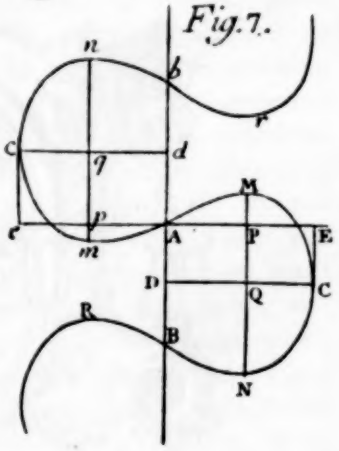


Fig.7.



Additamentum.

Tabula. IV.

Fig. 8.

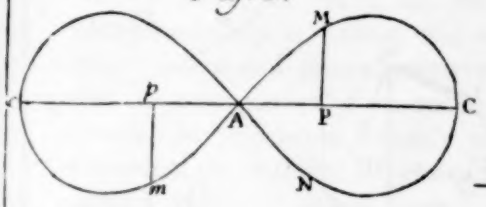


Fig. 9.

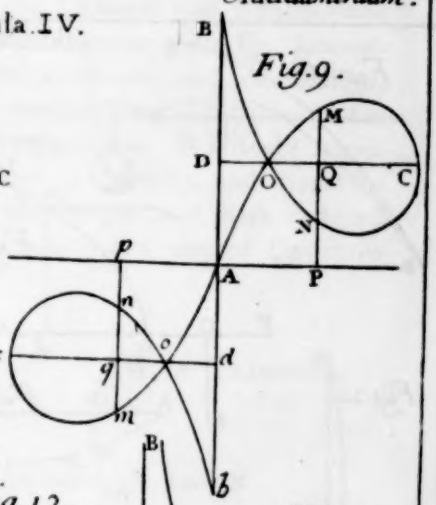


Fig. 11.

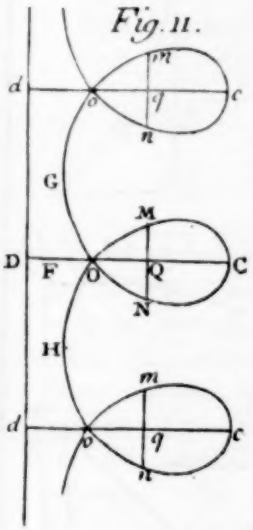


Fig. 12.

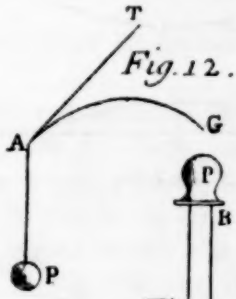


Fig. 10.

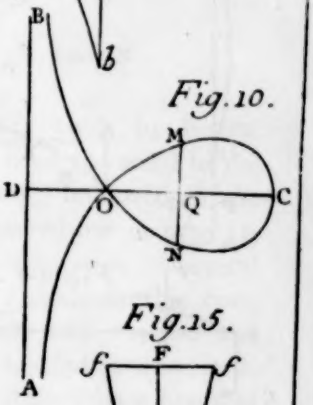


Fig. 13.

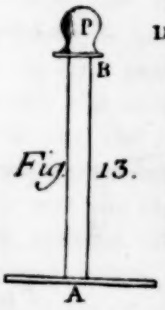


Fig. 15.

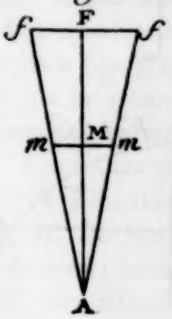
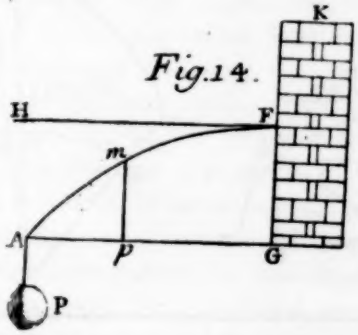
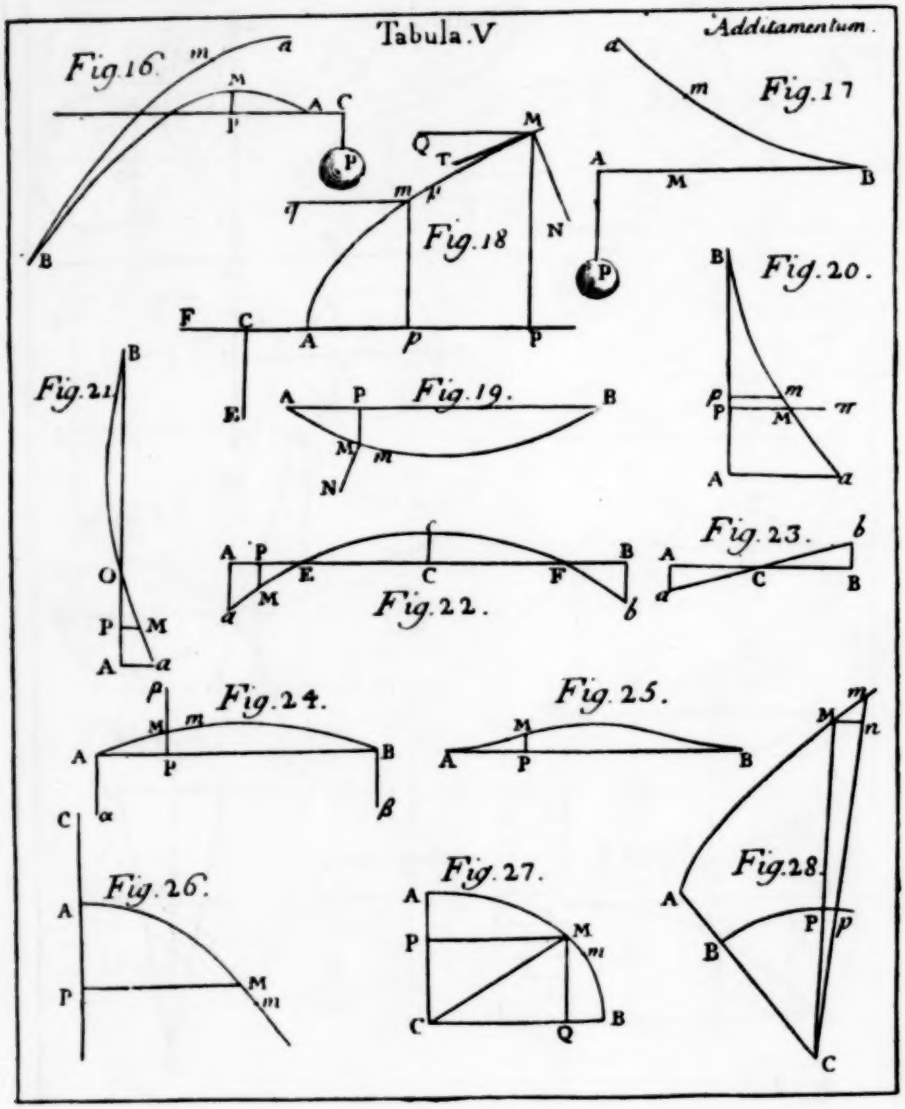


Fig. 14.



Tabula V

Additamentum.



very competent assistance of Mr. DONALD M. BROWN, formerly a student of Engineering, but at present an assistant in the Department of Mathematics of the University of Illinois, who undertook to revise the translation, together with Dr. LINSENBARTH's notes, to check all the equations and calculations, and occasionally to express the mathematical formulas in the more modern and generally current notation. It is to be hoped that the combination of an engineer, a classicist, and a mathematician in translating EULER's monograph may have reduced somewhat the number of errors which any one of the three unaided might easily have made.

Urbana, Illinois, April 27, 1932.

W. A. OLDFATHER.

ADDITIONAL NOTE BY DONALD M. BROWN

Such factual errors and mistakes as were made by EULER, and have been pointed out by others, have been corrected in the body of the text, the errors themselves being indicated in the notes. An exception to this is the error pointed out in note 31, where the correction would involve the incorporation of several sections of the body of the text into the notes. In this case, since the error involved was not great enough to make any essential difference in the calculations, the text was included as translated, and the correct values have been inserted within square brackets immediately following the incorrect values at all places where errors had been made. With the exception of the error indicated in note 23, all the errors were pointed out by H. LINSENBARTH in his German translation of the text in "Ostwald's Klassiker der exakten Wissenschaften," vol. 175 (Leipzig, 1910). In fact, all the notes, correctional, explanatory, and introductory, have been incorporated as translated, but several of the correctional notes have been modified to conform to the plan mentioned above of correcting the text, and indicating the errors themselves in the notes.

LINSENBARTH's admirable translation was used throughout as a check, and since this work includes numerous cross references

with the text which are not found in the original, such references have been inserted within square brackets.

The facsimiles of the title page and of the figures are from the Harvard Library copy; they are reduced to about two-thirds of the original size; Figs. 26, 27, and 28 belong to Additamentum II, *De motu projectorum in medio non resistente*.

Nothing has been omitted in this translation.

D. M. BROWN.

INTRODUCTION

"It is of the utmost importance," writes Professor G. A. MILLER (1), "that those students who desire a deep mathematical insight should accustom themselves early to go directly to the original developments,—at least in those cases where the original developments are direct, and are found in a language which does not impose too great difficulties." The statement might be equally applicable to the same kind of student of engineering, and it is for such students of mathematics and of engineering that the present translation from the original Latin into English is intended.

LEONHARD EULER (1707-1783), probably the most versatile, certainly quite the most prolific mathematician of all time (2), needs no commendation from us. CONDORCET, in his celebrated *Éloge* (3), after remarking that "all the celebrated mathematicians of today are his pupils," quite justly lists him as "one of the

(1) *Historical Introduction to Mathematical Literature* (New York, 1916) 74.

(2) By early in 1783, 530 studies of his had been published; by 1826 the number had increased to 771. In 1844 a great-grandson discovered still further material in manuscript. (G. DU PASQUIER: *Léonard Euler et ses amis* (Paris, 1927) 116). His complete bibliography, by G. ENESTRÖM: *Verzeichnis der Schriften Leonhard Eulers* (Leipzig, 1910 and 1913), lists 866 separate items, together with a voluminous correspondence. The *Leonhard Euler-Gesellschaft*, a society organized for the sole purpose of publishing his works in proper modern form, produced its first volume in 1911, and down to date has brought out some 20 volumes. The completed undertaking will require 69 quarto volumes, of which 55 are assigned to Mathematics, Mechanics, and Astronomy, the remainder to Physics and Varia.

(3) Published in *Les Lettres de L. Euler à une Princesse d'Allemagne* (Paris, 1842), xlviii and 1.

very greatest and most extraordinary men that nature has ever produced."

The treatise also to which the present study is merely an appendix *(Methodus Inveniendi Lineas Curvas)* is called by DU PASQUIER "one of the finest monuments of the genius of EULER," who, he continues, "founded the calculus of variations which has become, in the twentieth century, one of the most efficient of the means of investigation employed by mathematicians and physicists. The recent theories of EINSTEIN and the applications of the principle of relativity have greatly increased the importance of the calculus of variations which EULER created" (*op. cit.*, 50-51).

The special interest for engineering in the present little paper, lies in the fact that it is the first systematic treatment of elastic curves, laying the foundation for subsequent studies, and of course most immediately for the celebrated 'Euler formula,' "which expresses the critical load at which a slender column buckles." (4)

Referring to the *Additamentum I*, TODHUNTER and PEARSON say, "Euler distinguishes the various species of curves included under the general differential equation...

$$dy = \frac{(\alpha + \beta x + \gamma x^2)\, dx}{\sqrt{a^4 - (\alpha + \beta x + \gamma x^2)^2}}$$ [p. 82 of this translation.]

... The whole discussion is worthy of this great master of analysis;" (5) Again, "From page 282 [p. 121 of this translation] to the end EULER devotes his attention to the *oscillations* of an elastic lamina; the investigation is somewhat obscure for the science of dynamics had not yet been placed on the firm foundation of *D'Alembert's Principle:* Nevertheless, the results obtained by EULER will be found in substantial agreement with those in POISSON's *Traité de Mécanique*, Vol. II, pages 368-392. The important equations (a) and (a') on POISSON's pages 377 and 387 respectively agree with corresponding equations on EULER's" [pages 297 and 287 = pages 135 and 125 respectively of this translation] (6).

---

(4) H. M. WESTERGARD: *One Hundred and Fifty Years Advance in Structural Analysis. Transactions American Society of Civil Engineers* 94 (1930) 228. Compare also remarks by S. TIMOSHENKO (*ibid.* 241).

(5) TODHUNTER and PEARSON: *A History of the Theory of Elasticity and of the Strength of Materials.* Cambridge, University Press, 1886, p. 36.

(6) *Op. cit.*, p. 38.

Referring to EULER's *Sur la force des colonnes, Mémoires de l'Académie de Berlin*, Tom. XIII, 1759, pages 252-282, TODHUNTER and PEARSON (7) say, " This is one of EULER's most important contributions to the theory of elasticity.

The problem with which this memoir is concerned, is the discovery of the least force which will suffice to give the least curvature to a column, when applied at one extremity parallel to its axis, the other extremity being fixed. EULER finds that the force must be at least $\pi^2 \frac{Ek^2}{a^2}$, where a is the length of the column and Ek^2 is the ' moment of the spring,' or the ' moment of stiffness of the column'..." The volume from which this is quoted is intended to be a chronological development of the theory of elasticity, but the authors obviously overlooked the fact that the problem stated is precisely the one considered in section 37, pages 267-268 of the original *Additamentum I* [pages 102-103 of this translation]. Hence the present work is the first known source of the famous ' EULER formula ' mentioned above.

It might, in conclusion, be observed that the problem of elastic curves and the buckling of long thin struts under thrust was first worked out by EULER (8).

W. A. OLDFATHER and D. M. BROWN.

CONCERNING ELASTIC CURVES

L. EULER

1. All the greatest mathematicians have long since recognized that the method presented in this book is not only extremely useful in analysis, but that it also contributes greatly to the solution of physical problems. For since the fabric of the universe is most perfect, and is the work of a most wise Creator, nothing whatsoever takes place in the universe in which some relation of maximum and minimum does not appear. Wherefore there

(7) *Op. cit.*, p. 39.

(8) LOVE : *The Mathematical Theory of Elasticity*. Cambridge, University Press, Third Edition, 1920. Footnotes p. 407 and p. 411. See also Introduction, p. 3.

is absolutely no doubt that every effect in the universe can be explained as satisfactorily from final causes, by the aid of the method of maxima and minima, as it can from the effective causes themselves. Now there exist on every hand such notable instances of this fact, that, in order to prove its truth, we have no need at all of a number of examples; nay rather one's task should be this, namely, in any field of Natural Science whatsoever to study that quantity which takes on a maximum or a minimum value, an occupation that seems to belong to philosophy rather than to mathematics. Since, therefore, two methods of studying effects in Nature lie open to us, one by means of effective causes, which is commonly called the direct method, the other by means of final causes, the mathematician uses each with equal success. Of course, when the effective causes are too obscure, but the final causes are more readily ascertained, the problem is commonly solved by the indirect method; on the contrary, however, the direct method is employed whenever it is possible to determine the effect from the effective causes. But one ought to make a special effort to see that both ways of approach to the solution of the problem be laid open; for thus not only is one solution greatly strengthened by the other, but, more than that, from the agreement between the two solutions we secure the very highest satisfaction. Thus the curvature of a rope or of a chain in suspension has been discovered by both methods; first, *a priori*, from the attractions of gravity; and second, by the method of maxima and minima, since it was recognized that a rope of that kind ought to assume a curvature whose center of gravity was at the lowest point. Similarly, the curvature of rays passing through a transparent medium of varying density has been determined both *a priori*, and also from the principle that they ought to arrive at a given point in the shortest time.

Other similar examples have been brought forward in large numbers by the most eminent BERNOULLIS and others, who have made very great contributions both to the method of *a priori* solution, and to the knowledge of effective causes. Although, therefore, thanks to these so numerous and striking instances, there can be no doubt that in the case of all curved lines which appear in the solution of physical-mathematical problems, there enters in the character of some maximum or minimum; still this

very maximum or minimum is frequently very hard to recognize, although one might have reached a solution *a priori*. Thus, although the figure which a curved elastic ribbon assumes has long since been known, nevertheless no one has observed as yet how this curve can be studied by the method of maxima and minima, that is to say, by means of final causes. Wherefore, seeing that the most illustrious and, in this sublime fashion of studying nature, most perspicacious man, DANIEL BERNOULLI, had pointed out to me that he could express in a single formula, which he calls the potential force, the whole force which inheres in a curved elastic ribbon, and that this expression must be a minimum in the elastic curve [1]), and since by this discovery my method of maxima and minima as set forth in this book has had new light cast upon it in a marvelous fashion, and its most extensive application is thoroughly established, I cannot let pass this most desired opportunity without making clearer the application of my method at the same time that I publish this remarkable characteristic of the elastic curve discovered by the celebrated BERNOULLI. For that characteristic contains within itself differentials of the second order in such a fashion that the methods hitherto published of solving the isoperimetric problem are not capable of disclosing it.

ON THE CURVATURE OF UNIFORM ELASTIC RIBBONS

2. Let the elastic ribbon AB (FIG. 1) be curved in any direction whatsoever; let the arc $AM = s$, and the radius of curvature $MR = R$; furthermore, according to BERNOULLI, let the potential force contained in the portion AM of the ribbon be designated by the expression $\int \frac{ds}{R^2}$. If the ribbon be of uniform cross section and elasticity, and if it be straight when in its natural position, [2]) the character of the curve AM will be such that in this case the expression $\int \frac{ds}{R^2}$ is an absolute minimum. But since the differentials of the second order appear in the radius of curvature R, in order to determine a curve of this character we shall need four conditions, and this is precisely the

subject of our investigation. For since through the given ends A and B, an infinite number of elastic ribbons of the same length can be bent, the problem will not have been solved unless, in addition to the two points A and B, two other points, or what amounts to the same thing, the position of the tangents at the points A and B be given at the same time. For given an elastic ribbon which is longer than this distance between A and B, it can be curved not only in such a way that it is contained between the ends A and B, but also in such a way that its tangents have a given direction through these points. Wherefore the problem of finding the curvature of an elastic ribbon, which is to be solved by this method, must be expressed thus :

That among all curves of the same length, which not only pass through the points A and B, but also are tangent to given straight lines at these points, that curve be determined in which the value of $\int \frac{ds}{R^2}$ is a minimum.

3. Because the solution is to be referred to rectangular coordinates, let any straight line AD be taken as an axis, the abscissa $AP = x$, (FIG. 2) and the ordinate $PM = y$; then, according to this method, letting $dy = pdx$, and $dp = qdx$, the element of the curve Mm will be $ds = dx\sqrt{1 + p^2}$. Since the curves from among which the curve sought has to be discovered are to be isoperimetric, in the first place, the expression $dx\sqrt{1 + p^2}$ will have to be considered; this, compared with the general expression $\int Zdx$, gives the differential value $\frac{d}{dx}\frac{p}{\sqrt{1 + p^2}}$. Secondly, since the radius of curvature is $\frac{dx\ (1 + p^2)}{dp} = \frac{(1 + p_2)^{\frac{3}{2}}}{q} = R$, the expression $\int \frac{ds}{R^2}$ which must be a minimum, is transformed into $\int \frac{q^2\ dx}{(1 + p_2)^{\frac{3}{2}}}$. Let this be compared with the general expression $\int Zdx$, and this gives $Z = \frac{q^2}{(1 + p_2)^{\frac{3}{2}}}$, and letting $dZ = Mdx + Ndy + Pdp + Qdq$, then $M = o$,

$N = o$, $P = \frac{-5pq^2}{(1+p^2)^{\frac{7}{2}}}$, and $Q = \frac{2q}{(1+p^2)^{\frac{5}{2}}}$. Therefore the differential value to be derived from the expression $\int \frac{q^2\,dx}{(1+p^2)^{\frac{5}{2}}}$ will be $-\frac{dP}{dx} + \frac{d^2Q}{dx^2}$. And so for the curve sought we shall have the equation :

$$a\,\frac{d}{dx}\,\frac{p}{\sqrt{1+p^2}} = \frac{dP}{dx} - \frac{d^2Q}{dx^2}$$

which, multiplied by dx and integrated, gives

$$\frac{ap}{\sqrt{1+p^2}} + \beta = P - \frac{dQ}{dx}$$

Let this equation be multiplied by $qdx = dp$.

$$\frac{apdp}{\sqrt{1+p^2}} + \beta dp = Pdp - qdQ$$

Since $M = o$, and $N = o$, then $dZ = Pdp + Qdq$, or $Pdp = dZ - Qdq$.

Substituting this value for Pdp gives

$$\frac{apdp}{\sqrt{1+p^2}} + \beta dp = dZ - Qdq - qdQ$$

Integrating :

$$a\sqrt{1+p^2} + \beta p + \gamma = Z - Qq$$

Now since

$$Z = \frac{q^2}{(1+p^2)^{\frac{5}{2}}} \text{ and } Q = \frac{2q}{(1+p^2)^{\frac{5}{2}}}$$

$$a\sqrt{1+p^2} + \beta p + \gamma = \frac{-q^2}{(1+p^2)^{\frac{5}{2}}}$$

Taking the arbitrary constants negatively,

$$q = (1+p^2)^{\frac{5}{4}} \sqrt{a\sqrt{1+p^2} + \beta p + \gamma} = \frac{dp}{dx}$$

whence

$$dx = \frac{dp}{(1+p^2)^{\frac{5}{4}} \sqrt{a\sqrt{1+p^2} + \beta p + \gamma}}$$

Then, since $dy = pdx$,

$$dy = \frac{pdp}{(1+p^2)^{\frac{1}{2}} \sqrt{\alpha \sqrt{1+p^2} + \beta p + \gamma}}$$

These two equations would be sufficient for constructing a curve by means of quadratures.

4. Neither of these equations regarded thus in general can be integrated, but they can be combined in a certain fashion so that the sum can be integrated. For, since

$$d2\frac{\sqrt{\alpha \sqrt{1+p^2} + \beta p + \gamma}}{\sqrt[4]{1+p^2}} = \frac{dp\ (\beta - \gamma p)}{(1+p^2)^{\frac{1}{2}} \sqrt{\alpha \sqrt{1+p^2}+\beta p+\gamma}},$$

then $2\frac{\sqrt{\alpha \sqrt{1+p^2}+\beta p+\gamma}}{(1+p^2)^{\frac{1}{4}}} = \beta x - \gamma y + \delta.$

Since the position of the axis is arbitrary, the constant δ can be left out without any loss in generality. Moreover, the axis can be so changed that the abscissa will become $X = \frac{\beta x - \gamma y}{\sqrt{\beta^2 + \gamma^2}}$, and the ordinate will become $Y = \frac{\gamma x + \beta y}{\sqrt{\beta^2 + \gamma^2}}$. [3]) Also γ can be safely made equal to zero, because nothing prevents the new abscissa from being expressed by x. For this reason we will get the following equation for the elastic curve:

$$2\sqrt{\alpha \sqrt{1+p^2} + \beta p} = \beta x\ (1+p^2)^{\frac{1}{4}},$$

which, after squaring becomes $4\alpha \sqrt{1+p^2} + 4\beta p = \beta^2 x^2 \sqrt{1+p^2}$.

To introduce homogeneity, let $\alpha = \frac{4m}{a^2}$, and $\beta = \frac{4n}{a^2}$;

then $na^2 p = (n^2x^2 - ma^2)\sqrt{1+p^2}$,
whence $n^2a^4p^2 = (n^2x^2 - ma^2)^2\ (1 + p^2)$,
and therefore

$$p = \frac{n^2\ x^2 - ma^2}{\sqrt{n^2\ a^4 - (n^2\ x^2 - ma^2)^2}}.$$

By changing the constants, and either by increasing or diminishing

the abscissa x by a given constant, [4]) the following general equation for the elastic curve will be secured:

$$dy = \frac{(\alpha + \beta x + \gamma x^2)\, dx}{\sqrt{a^4 - (\alpha + \beta x + \gamma x^2)^2}},$$

from which

$$ds = \frac{a^2 dx}{\sqrt{a^4 - (\alpha + \beta x + \gamma x^2)^2}}.$$

From these equations, the agreement of this discovered curve with the elastic curve already determined is perfectly clear.

5. In order that this agreement be placed more clearly before the reader, I shall investigate also *a priori* the nature of the elastic curve. Although this has been done already in a most excellent fashion by that very great man, JACOB BERNOULLI, nevertheless, since this favorable occasion has been offered, I shall add certain things about the character of elastic curves and their different kinds and figures which I see have been either neglected by other men, or else have been but lightly touched upon.

Let the elastic ribbon AB (FIG. 3) be fixed in a wall or solid pavement at B in such a fashion that the extremity B is not only held firmly, but also the position of the tangent at B is fixed. Now at A let the ribbon have fastened to it the rigid rod AC, to which let there be applied normally the force $CD = P$, whereby the ribbon is brought into the curved position BMA. Let this straight line AC be considered as produced for an axis, and, having assumed that $AC = c$, let the abscissa $AP = x$, and the ordinate $PM = y$. If now the ribbon at M should suddenly lose all elasticity and become perfectly flexible, it would assuredly be turned by the force P, the inflexion being caused by the moment of the force $P = P(c + x)$. The reason why this inflexion does not actually follow, therefore, is that the elasticity depends, in the first place, upon the character of the material of which the ribbon is composed and which I assume always to be the same; but in the second place the elasticity depends, at the same time, upon the curvature of the ribbon at the point M, in such a way that it is inversely proportional to the radius of curvature at M. Therefore let the radius of curvature

at M be $R = \frac{(ds)^3}{-dxd^2y}$; here $ds = \sqrt{dx^2 + dy^2}$, and dx remains constant; and let $\frac{Ek^2}{R}$ express the elastic force of the ribbon at M, which stands in equilibrium with the moment of the external force $P(c + x)$, in such a manner that

$$P\,(c + x) = \frac{Ek^2}{R} = -\frac{Ek^2dxd^2y}{(ds)^3}.$$

This equation, multiplied by dx, becomes integrable, and the integral will be

$$P\,(\tfrac{1}{2}x^2 + cx + f) = \frac{-Ek^2dy}{\sqrt{dx^2 + dy^2}},$$

whence

$$dy = \frac{-Pdx\,(\tfrac{1}{2}x^2 + cx + f)}{\sqrt{E^2k^4 - P^2\,(\tfrac{1}{2}x^2 + cx + f)^2}}.$$

This equation agrees absolutely with that which I have just secured through the method of maxima and minima from Bernoulli's principle.

6. From the comparison of this equation with the one found before, it will be possible to determine the force which is required to produce the given curvature of the ribbon, since the curvature is contained in the discovered general equation. In other words, let the elastic ribbon have the shape AMB, the nature of which is expressed by the equation

$$dy = \frac{(\alpha + \beta x + \gamma x^2)\,dx}{\sqrt{a^4 - (\alpha + \beta x + \gamma x^2)^2}},$$

and let Ek^2 express the absolute elasticity of this ribbon in such a manner, indeed, that Ek^2 at any point you please, divided by the radius of curvature, represents the true elastic force. To institute a comparison, let the numerator and denominator be each multiplied by $\frac{Ek^2}{a^2}$, so that we have

$$dy = \frac{\frac{Ek^2\,dx\,(\alpha + \beta x + \gamma x^2)}{a^2}}{\sqrt{E^2k^4 - \frac{E^2k^4}{a^4}\,(\alpha + \beta x + \gamma x^2)^2}}$$

Therefore

$$-\tfrac{1}{2}P = \frac{Ek^2\gamma}{a^2}; \quad -PC = \frac{Ek^2\beta}{a^2}; \quad \text{and} \quad -Pf = \frac{Ek^2\alpha}{a^2};$$

and hence the external force $CD = \dfrac{-2Ek^2\gamma}{a^2}$, the length $AC = c = \dfrac{\beta}{2\gamma}$, and the constant $f = \dfrac{\alpha}{2\gamma}$.

7. In order, therefore, that the elastic ribbon AB, fixed in the wall at one end B, be curved into the shape AMB, the character of which is expressed by the following equation :

$$dy = \frac{(\alpha + \beta x + \gamma x^2)\,dx}{\sqrt{a^4 - (\alpha + \beta x + \gamma x^2)^2}},$$

it is necessary that the ribbon be drawn in the direction CD normal to the axis AP (assuming that the distance $AC = \dfrac{\beta}{2\gamma}$), by the force $CD = \dfrac{-2Ek^2\gamma}{a^2}$. This force will act, of course, in an opposite direction from that shown in the figure, if γ be a positive quantity. Because $\dfrac{Ek^2}{R}$ is equivalent to the moment of the external force, the expression $\dfrac{Ek^2}{a^2}$ will be equivalent to the weight, or to a pure force, which force, $\dfrac{Ek^2}{a^2}$, for that reason, will be determined by the elasticity of the ribbon. Let this force be F; then the deflecting force CD will be to this force F as -2γ is to 1, for γ will be an abstract number.

8. Now in addition, the force required to keep the portion BM of the ribbon in its position, if the portion AM should be entirely cut off, can be determined from this. When this portion AM is cut off, the elastic ribbon becomes a rigid rod MT [FIG. 3] without any flexure at all, and this is so connected with the ribbon that it always makes a tangent at the point M, no matter how the ribbon may be bent. If we assume this, it is clear from what precedes that to preserve the curvature BM, it is necessary

that the rod MT at the point N be drawn in the direction ND by a force which equals $-\frac{2Ek^2\gamma}{a^2}$; the direction ND will be normal to the axis AP, and likewise the interval AC will be equal to $\frac{\beta}{2\gamma}$. And thus the distance MN will be

$$\frac{ds}{dx}.\,CP = \frac{ds}{dx}.\frac{\beta+2\gamma x}{2\gamma} = \frac{(\beta+2\gamma x)\,ds}{2\gamma\,dx}.$$

Then

$$\frac{ds}{dx} = \frac{a^2}{\sqrt{a^4-(\alpha+\beta x+\gamma x^2)^2}}$$

But if this force ND, which equals $-\frac{2Ek^2\gamma}{a^2}$, be resolved into two components, NQ normal to the tangent MT, and NT along MT, the normal force $NQ = \frac{-2Ek^2\gamma}{a^2}.\frac{dx}{ds}$, and the tangential force $NT = \frac{-2Ek^2\gamma}{a^2}.\frac{dy}{ds}$.

9. But now if the part BM be cut off, leaving the part AM, which is drawn as before in the direction CD by the force $-\frac{2Ek^2\gamma}{a^2}$, in order to preserve the curvature AM, the extremity M, which is understood to be connected with the rigid tangent rod MN, will have to be drawn, at the point N, by a force also equal to $\frac{-2Ek^2\gamma}{a^2}$, but in the direction opposite to that which we have discovered in the preceding case. For the forces which will have to be applied to both extremeties of the curved ribbon constantly oppose each other, and consequently must be equal and opposite. For otherwise the whole ribbon would be moved, and to restrain this movement, a force would be necessary to cause equilibrium between these forces. Hence the forces to be applied at any portion of the part which has been cut off can be determined; these forces will preserve the curvature already induced.

10. Let AM (FIG. 4) be an elastic ribbon, to which, at A and M are attached rigid rods AD and MN, and to which in opposite directions DE and NR there have been applied equal forces DE and NR, which, being in equilibrium, induce the curvature AM in the ribbon. For this curvature an equation will be sought. First, therefore, let there be taken as an axis the straight line AP passing through the point A, and normal to the direction ER of the external force. Let the absolute elasticity of the ribbon be Ek^2; and let the sine of the angle CAD which the tangent forms with the axis at A, and which has been given, equal m, and the cosine equal n, so that $m^2 + n^2 = 1$. Furthermore, let the distance $AC = c$, and the bending force $DE = NR = P$. Letting the abscissa $AP = x$, and the ordinate $PM = y$, the character of the curve will be expressed by the following equation :

$$dy = \frac{-Pdx\,(\frac{1}{2}x^2 + cx + f)}{\sqrt{E^2k^4 - P^2\,(\frac{1}{2}x^2 + cx + f)^2}}$$

But since the direction of the tangent at A is given, when $x = o$, $\frac{dy}{dx} = \frac{m}{n}$ whence $\frac{m}{n} = \frac{-Pf}{\sqrt{E^2k^4 - P^2f^2}} = \frac{m}{\sqrt{1-m^2}}$, and $m = \frac{-Pf}{Ek^2}$. Therefore the constant f is determined, so that $f = \frac{-mEk^2}{P}$, and hence the whole curve is thereby determined.

11. To produce, therefore, the curvature of the ribbon AM, expressed by the foregoing equation, the force $DE = P$ must be applied to the tangent AD, at the point D, in such a way that $AD = \frac{c}{n}$, and in a direction parallel to the ordinate PM. Let this force DE be resolved into two rectangular components Dd and Df (FIG. 5), normal to one another, the force $Dd = Pn$, and the force $Df = Pm$. Now in order that the consideration of the straight line AD may be eliminated from the computation, in place of the force Dd at the given points A and B (assuming that $AB = h$), two forces $Aa = p$ and $Bb = q$ can be substituted, likewise normal to the rod AB, if we let $Ph = Pn.BD = nP(\frac{c}{n} - h)$, and $q = p + nP$. In the next place, because it makes no differ-

ence at which point of the rod AB the tangential force $Df = mP$ be applied, let it be applied at the point A, where $AF = mP$. Now let the force $AF = r$ in such a way that the ribbon MA is acted upon by the three forces $Aa = p$, $Bb = q$, and $AF = r$. We shall investigate how the curvature is affected by them.

12. First, since $mP = r$, $P = \frac{r}{m}$, which value, substituted in the former equations, will give $ph = \frac{cr}{m} - \frac{nhr}{m}$, and $q = p + \frac{nr}{m}$; hence $\frac{n}{m} = \frac{q - p}{r}$ from which equation first the position of the axis AP becomes known; for $\tan CAD = \frac{r}{q - p}$.

Hence

$$m = \frac{r}{\sqrt{r^2 + (q-p)^2}}, \text{ and } n = \frac{q-p}{\sqrt{r^2 + (q-p)^2}}.$$

Secondly, from the equation $hp = \frac{cr}{m} - \frac{nhr}{m} = \frac{cr}{m} - hq + hp$, it follows that $c = \frac{mhq}{r}$, or $c = \frac{hq}{\sqrt{r^2 + (q-p)^2}}$, and

$$P = \sqrt{r^2 + (q-p)^2}.$$

Now since

$$f = \frac{-mEk^2}{P} = \frac{-Ek^2r}{r^2 + (q-p)^2},$$

then

$$\frac{x^2}{2} + cx + f = \frac{x^2}{2} + \frac{hqx}{r^2 + (q-p)^2} - \frac{Ek^2r}{r^2 + (q-p)^2},$$

from which the following equation of the curve sought will be obtained :

$$dy = \frac{dx \left[\frac{Ek^2r}{\sqrt{r^2 + (q-p)^2}} - hqx - \frac{1}{2} x^2 \sqrt{r^2 + (q-p)^2} \right]}{\sqrt{E^2k^4 - \left[\frac{Ek^2r}{\sqrt{r^2 + (q-p)^2}} - hqx - \frac{1}{2} x^2 \sqrt{r^2 + (q-p)^2} \right]^2}}$$

Now this equation is very convenient for the most common method of bending ribbons when they are held either by forceps or two fingers, one of which presses in a direction *Aa*, the other in the direction *Bb*, while, at the same time, the ribbon can be stretched in the direction *AF*.

13. If the tangential force $AF = r$ should disappear, the axis AP will fall upon the tangent AF produced, and

$$dy = \frac{-dx\,[hqx + \frac{1}{2}(q-p)\,x^2]}{\sqrt{E^2k^4 - [hqx + \frac{1}{2}(q-p)\,x^2]^2}}.$$

But if the normal forces p and q should be equal, the axis AP will be normal to the tangent AF, because $n = o$, and we shall have the following equation for the curve:

$$dy = \frac{dx\,(Ek^2 - hqx - \frac{1}{2}rx^2)}{\sqrt{2\,Ek^2\,(hqx + \frac{1}{2}rx^2) - (hqx + \frac{1}{2}rx^2)^2}}.$$

Hence if also $r = o$ in such a way that the ribbon at the points A and B be subjected to equal and opposite forces Aa and Bb, the character of the curve will be expressed by

$$dy = \frac{dx\,(Ek^2 - hqx)}{\sqrt{hq\,(2Ek^2x - hqx^2)}},$$

which, when integrated, gives

$$y = \int \frac{\sqrt{2Ek^2x - hqx^2}}{hq}.$$

This is the equation of a circle, and therefore, in this case, the ribbon is bent into the arc of a circle, the radius of which will be $\frac{Ek^2}{hq}$.

The Enumeration of Elastic Curves

14. Since therefore we observe that not only is the circle included in the class of elastic curves, but more than that, there is an infinite variety of these elastic curves, it will be worth while to enumerate all the different kinds included in this class of curves. For in this way not only will the character of these curves be more profoundly perceived, but also, in any case whatsoever

offered, it will be possible to decide from the mere figure into what class the curve formed ought to be put. We shall also list here the different kinds of curves in the same way in which the kinds of algebraic curves included in a given order are commonly enumerated [5]).

15. The general equation for elastic curves is

$$dy = \frac{(\alpha + \beta x + \gamma x^2)\, dx}{\sqrt{a^4 - (\alpha + \beta x + \gamma x^2)^2}},$$

which, if the origin of the abscissas be moved on the axis through the distance $\frac{\beta}{2\gamma}$, and if a^2 be written for $\frac{a^2}{\gamma}$ (or making $\gamma = 1$), takes the simpler form

$$dy = \frac{(\alpha + x^2)\, dx}{\sqrt{a^4 - (\alpha + x^2)^2}}.$$

But because $a^4 - (\alpha + x^2)^2 = (a^2 - \alpha - x^2)\,(a^2 + \alpha + x^2)$, let $a^2 - \alpha = c^2$, so that $\alpha = a^2 - c^2$, and the equation will be transformed into

$$dy = \frac{(a^2 - c^2 + x^2)\, dx}{\sqrt{(c^2 - x^2)\,(2a^2 - c^2 + x^2)}}.$$

Let the character of the curve AMC (FIG. 6) be expressed by this equation, and the abscissa $AP = x$, and the ordinate $PM = y$. Therefore, since $\beta = 0$ [see FIG. 3, Sec. 6], the direction of the force which bends the elastic ribbon will be normal to the axis AP at the point A, and therefore AD will represent the direction of the acting force. This force will equal $\frac{2Ek^2}{a^2}$ where Ek^2 expresses the absolute elasticity.

16. If x = 0, then $\frac{dy}{dx} = \frac{a^2 - c^2}{c\sqrt{2a^2 - c^2}}$. This expression gives the tangent of the angle which the curve AM makes with the axis AP at A, the sine of which angle will equal $\frac{a^2 - c^2}{a^2}$. Wherefore, if $a^2 = \infty$, the ribbon will be normal to the axis AP at the point A, and will have no curvature, because the curving

force $\frac{2Ek^2}{a^2}$ disappears. Therefore, in the case where $a^2 = \infty$, the natural shape of the ribbon appears, that is, a straight line. This, then, constitutes the first class of elastic curves, which the straight line AB produced in both directions to infinity will represent.

17. Before enumerating the remaining classes, it will be convenient to make certain observations in general about the figure of the elastic curve. Now it is understood that the angle PAM [FIG. 6] which the curve makes with the axis AP at A, decreases as the quantity a^2 becomes smaller, that is, the more the curving force $\frac{2Ek^2}{a^2}$ is applied. And if a^2 should become equal to c^2, then the axis AP will be tangent to the curve at A; but if $a^2 < c^2$, then the curve AM, which hitherto ran downwards [as in FIG. 6], will now turn upwards until [as in FIG. 7] the point is reached where $a^2 = \frac{c^2}{2}$, in which case the tangent of the curve will fall upon the straight line Ab. But if $a^2 < \frac{1}{2}c^2$, then the angle PAM will be absolutely imaginary, and therefore no portion of the curve will exist at A. These different cases will constitute a variety of classes.

18. Furthermore from the equation it is understood (because if x and y are both made negative, the form of the equation is not changed) that the curve on both sides of A has similar and equal branches AMC and Amc alternately disposed, in such a way that A is the point of contraflexure; whence, the portion AMC of the curve being known at the same time, its continuation Amc beyond A will be known, in as much as the latter is similar and equal to the former. Thus, letting $Ap = AP$, pm will also equal PM. Now in receding from A, the curve on both sides is bent back further from the axis, until the abscissa $AE = c$, the ordinate EC will be tangent to the curve; for if $x = c$, then $\frac{dy}{dx} = \infty$. It is clear that the abscissa x cannot increase beyond $AE = c$, for otherwise $\frac{dy}{dx}$ would become imaginary. Hence

the whole curve will be contained between the extreme ordinates EC and ec, beyond which limits it cannot pass. Now therefore we have, so far, the two branches AC and Ac of the curve extending on both sides from A to the limits.

19. Let us see, then, under what conditions the curve may pass beyond C and c. To this end let us take the straight line CD parallel to AE as an axis, and let these new co-ordinates $CQ = t$ and $QM = u$; then $x + t = AE = CD = c$, and $y + u = CE = AD = b$, whence $x = c - t$, and $y = b - u$, or $dy = -du$, and $dx = -dt$. Substituting these values, there will arise an equation for the curve in terms of the new co-ordinates $CQ = t$ and $QM = u$; this equation will be

$$du = \frac{(a^2 - 2ct + t^2)\,dt}{\sqrt{t\,(2c - t)\,(2a^2 - 2ct + t^2)}}$$

Here it is clear, in the first place, that if t be taken as infinitely small, then $du = \frac{a^2 dt}{2a\sqrt{ct}}$, and $u = a\sqrt{\frac{t}{c}}$. The latter equation indicates that the curve beyond C begins to advance towards N in a way similar to that in which it extends from C to M [6]). Now the ambiguity of the radical sign in the denominator of the equation shows admirably that the ordinate u can be taken negatively as well as positively; whence it is manifest that the straight line CD is a diameter of the curve, and moreover, that the arc CNB will be similar and equal to the arc CMA.

20. Now in a similar way the straight line cd produced through c on the other side of and parallel to the axis AE will be a diameter of the curve; because the branch Acb is similar and equal to the branch ACB. Therefore at the points B and b there will also be points of contraflexure as at A; whence the curve will extend further in a similar fashion. Therefore the curve will have an infinite number of diameters CD, cd, etc., mutually distant from one another by the same interval Dd, and parallel to one another; and because of this, the curve will consist of an infinite number of parts similar and equal to one another; and therefore the whole curve will be known if only a single portion AMC be known.

21. Because the point of contraflexure is at A, the radius of curvature will be infinitely great at that point, which is clear from the nature of the curve. For since the curve at A is drawn by the force $\frac{2Ek^2}{a^2}$ in the direction AD, at any point M, if the radius of curvature be set equal to R, because of the nature of the elasticity, the force will be $\frac{2Ek^2x}{a^2} = \frac{Ek^2}{R}$, whence $R = \frac{a^2}{2x}$. Therefore at the point A ($x = o$) the radius of curvature is infinite; but because $AE = Ae = c$ at the points C and c, the radius of curvature will equal $\frac{a^2}{2c}$; in other words, at these places, the farthest distant from the straight line BAb, the curvature is greatest [7]).

22. Now although for the point C it is known that the abscissa is $AE = c$, nevertheless the distance EC cannot be determined except by the integration of the equation

$$dy = \frac{(a^2 - c^2 + x^2)\,dx}{\sqrt{(c^2 - x^2)(2a^2 - c^2 + x^2)}}.$$

For if after the integration, x be made equal to c, the value of y will give the distance CE, which taken twice will give the distance AB, or the interval Dd lying between the diameters. Similarly, integration will be necessary to determine the length of the curved ribbon AC. For since, if the arc $AM = s$,

$$ds = \frac{a^2\,dx}{\sqrt{(c^2 - x^2)(2a^2 - c^2 + x^2)}}, \text{ and}$$

its integral, evaluated at $x = c$, will give the length of the curve AC [8]).

23. Now since these formulas do not admit of integration, let us try to express conveniently by approximation the values of the interval AD and of the arc AC. To this end, let $\sqrt{c^2 - x^2} = z$, whence

$$PM. = y = \int \frac{(a^2 - z^2)\,dx}{z\sqrt{2a^2 - z^2}}, \text{ and } AM = s = \int \frac{a^2\,dx}{z\sqrt{2a^2 - z^2}}$$

Expressed as a series,

$$\frac{1}{\sqrt{2a^2-z^2}} = \frac{1}{a\sqrt{2}}\left(1 + \frac{1}{4}\cdot\frac{z^2}{a^2} + \frac{1.3}{4.8}\cdot\frac{z^4}{a^4} + \frac{1.3.5}{4.8.12}\cdot\frac{z^6}{a^6} + \ldots\right),$$

whence

$$s = \frac{1}{\sqrt{2}}\int\left(\frac{a}{z} + \frac{1}{4}\cdot\frac{z}{a} + \frac{1.3}{4.8}\cdot\frac{z^3}{a^3} + \frac{1.3.5}{4.8.12}\cdot\frac{z^5}{a^5} + \ldots\right)dx,$$

and

$$s-y = \frac{1}{\sqrt{2}}\int\left(\frac{z}{a} + \frac{1}{4}\cdot\frac{z^3}{a^3} + \frac{1.3}{4.8}\cdot\frac{z^5}{a^5} + \frac{1.3.5}{4.8.12}\cdot\frac{z^7}{a^7} + \ldots\right)dx.$$

24. Since we desire these integrals only for the case $x = c$, in which $z = o$, they can be expressed conveniently by the aid of the circumference of the circle. For assuming that the ratio of the diameter to the circumference is as 1 is to π,

$$\int\frac{dx}{z} = \int_o^c \frac{dx}{\sqrt{c^2-x^2}} = \frac{\pi}{2}.$$

Now in the same way the following integrals will be determined [9])

$$\int_o^c z\,dx = \frac{1}{2}\cdot\frac{\pi c^2}{2}, \quad \int_o^c z^3dx = \frac{1.3}{2.4}\cdot\frac{\pi}{2}c^4$$

$$\int_o^c z^5dx = \frac{1.3.5}{2.4.6}\cdot\frac{\pi}{2}c^6, \text{ and } \int_o^c z^7dx = \frac{1.3.5.7}{2.4.6.8}\cdot\frac{\pi}{2}c^8.$$

By the aid of these integrals we have

$$AC = \frac{\pi a}{2\sqrt{2}}\left(1 + \frac{1^2}{2^2}\cdot\frac{c^2}{2a^2} + \frac{1^2\,3^2}{2^2\,4^2}\cdot\frac{c^4}{4a^4} + \frac{1^2\,3^2\,5^2}{2^2\,4^2\,6^2}\cdot\frac{c^6}{8a^6} + \ldots.\right)$$

and

$$AD = \frac{\pi a}{2\sqrt{2}}\left(1 - \frac{1^2}{2^2}\cdot\frac{3}{1}\cdot\frac{c^2}{2a^2} - \frac{1^2\,3^2}{2^2\,4^2}\cdot\frac{5}{3}\cdot\frac{c^4}{4a^4} - \frac{1^2\,3^2\,5^2}{2^2\,4^2\,6^2}\cdot\frac{7}{5}\cdot\frac{c^6}{8a^6} - \ldots.\right)$$

If accordingly $AE = c$ and $AD = b$ be given, from these equations the constant a and the length of the curve AC will be determined. Conversely also, from the given length of this curve AC, and from the constant a by which the external force is determined, it will be possible to find the distances AD and CD.

First Class

25. Since we have so determined the first class that, in the general equation

$$dy = \frac{(a^2 - c^2 + x^2)\,dx}{\sqrt{(c^2 - x^2)\,(2a^2 - c^2 + x^2)}},$$

$c = o$, or $\frac{a}{c} = \infty$, a straight line represents the natural condition of the ribbon; and to this same first class let us refer also those cases for which c is an infinitely small quantity, in such a way that in comparison with a it can be regarded as on the point of vanishing. However, because x cannot be greater than c, likewise x in comparison with a will be on the point of vanishing, and therefore the following equation will result:

$$dy = \frac{a\,dx}{\sqrt{2\,(c^2 - x^2)}}.$$

The integral of this is $y = \frac{a}{\sqrt{2}} \text{ arc sin } \frac{x}{c}$, which is the equation for a trochoid curve infinitely elongated [10]). Now AD will become equal to $\frac{\pi a}{2\sqrt{2}}$, from which the length of the curve differs only infinitesimally, because the angle DAM is infinitely small. Let the length of the ribbon $ACB = 2f$, and its absolute elasticity be Ek^2. Because $f = \frac{\pi a}{2\sqrt{2}}$, the force requisite to produce this infinitely small curvature of the ribbon will be of a finite magnitude, and will equal $\frac{Ek^2}{f^2} \cdot \frac{\pi^2}{4}$; that is to say, if the extremeties A and B be fastened together with a string AB, the string will necessarily be stretched by the force $\frac{Ek^2}{f^2} \frac{\pi^2}{4}$.

Second Class

26. Let the case in which c is contained between the limits o and a, constitute the second class. In these cases the angle

DAM will be less than a right angle; for the sine of the angle PAM, or the cosine of the angle $DAM = \frac{a^2 - c^2}{a^2}$. Therefore in this case the form of the curve will be similar to that which FIG. 6 represents. Since $c < a$, therefore $\frac{c^2}{2a^2} < \frac{1}{2}$; but since $\frac{c^2}{2a^2} > o$, assuredly $AC = f > \frac{\pi a}{2\sqrt{2}}$, whence $a^2 < \frac{8f^2}{\pi^2}$; wherefore the force whereby the extremeties A and B of the ribbon are drawn together, by the aid of the string AB, will be greater than in the preceding case, that is the force will be greater than $\frac{Ek^2}{f^2} \cdot \frac{\pi^2}{4}$.

THIRD CLASS

27. In the third class I consider the unique case in which $c = a$, because in this case the axis AP is tangent to the curve at the point A. This class has the special name of the rectangular elastic curve. In this case

$$dy = \frac{x^2\,dx}{\sqrt{a^4 - x^4}}, \text{ and } ds = \frac{a^2 dx}{\sqrt{a^4 - x^4}},$$

and hence

$$AC = f = \frac{\pi a}{2\sqrt{2}}\left(1 + \frac{1^2}{2^2} \cdot \frac{1}{2} + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} \cdot \frac{1}{4} + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2} \cdot \frac{1}{8} + \ldots\right),$$

and

$$AD = b = \frac{\pi a}{2\sqrt{2}}\left(1 - \frac{1^2}{2^2} \cdot \frac{3}{1.2} - \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} \cdot \frac{5}{3 \cdot 4} - \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2} \cdot \frac{7}{5.8} - \ldots\right).$$

Now although from this neither b nor f can be accurately expressed in terms of a, yet I have elsewhere pointed out that there is a remarkable relation between these two quantities [11]). In other words I have shown that $4bf = \pi a^2$, or the rectangle formed by AD and AC will equal the area of a circle the diameter of which is AE. Now it will be found by introducing calculus that $f = \frac{5a}{6} \cdot \frac{\pi}{2}$ approximately, so that $a = \frac{12f}{5\pi}$; hence the force

by which the extremities A and B must be drawn toward each other will equal $\frac{Ek^2}{f^2}\cdot\frac{25}{72}\pi^2$. A closer approximation gives $f = 1.1803206\frac{a\pi}{2\sqrt{2}}$, hence

$b = \frac{\pi a^2}{4f} = \frac{a}{1.1803206\sqrt{2}}$, whence $\frac{f}{a} = 1.311006$, and $\frac{b}{a} = 0.59896$ [12]).

Fourth Class

28. If $c > a$, the fourth class will arise (Fig. 7), the ribbon opening out horizontally until $AD = b > o$. This second limit of c will be defined by the equation

$$1 = \frac{1^2}{2^2}\cdot\frac{3}{1}\cdot\frac{c^2}{2a^2} + \frac{1^2\cdot 3^2}{2^2\,4^2}\cdot\frac{5}{3}\cdot\frac{c^4}{4a^4} + \frac{1^2\cdot 3^2\cdot 5^2}{2^2\cdot 4^2\cdot 6^2}\cdot\frac{7}{5}\cdot\frac{c^6}{8a^6} + \ldots\ldots .$$

In this class therefore, since $c > a$, the curve at A will rise above the axis AE, and will form the angle PAM, the sine of which will equal $\frac{c^2 - a^2}{a^2}$. Now we shall soon see that this angle PAM is less than 40° -41′; since if it reaches this value, the interval AD disappears, a case which I refer to the fifth class. Hence in the fourth class are included the curves in which the value $\frac{c^2}{a^2}$ is contained between the limits 1 and 1.651868. Now the form of these curves is understood from the figure, provided only that it be observed that the closer $\frac{c^2}{a^2}$ approaches the latter limit 1.651868, the shorter the interval AD will become, and the closer the end points A and B will be brought to each other. Therefore it can happen that the humps of the ribbon m and R and likewise M and r are not merely mutually tangent, but even intersect, and intersections of this kind will be repeated indefinitely until all the diameters DC and dc coincide and merge with the axis AE.

Fifth Class

29. If this happens, the fifth class (Fig. 8) will arise, the character of which will be expressed by the equation between the co-ordinates $AP = x$ and $PM = y$,

$$dy = \frac{(a^2 - c^2 + x^2)\,dx}{\sqrt{(c^2 - x^2)\,(2a^2 - c^2 + x^2)}},$$

the following relation existing between a and c, viz., that the interval $AD = b = o$. Let $\frac{c^2}{2a^2} = v$, and then v must be defined by the following equation:

$$1 = \frac{1.3}{2.4}v + \frac{1.1.3.5}{2.2.4.4}v^2 + \frac{1.1.3.3.5.7}{2.2.4.4.6.6}v^3 + \ldots.$$

Let there be sought first, by methods familiar to everyone, or else by mere trial, the limits between which the true value of v is contained, and these limits will be found to be $v = 0.824$, and $v = 0.828$. But if now both of these values be substituted in the equation, from the two errors which are certain to arise, it will finally be concluded that $v = 0.825934 = \frac{c^2}{2a^2}$, whence $\frac{c^2}{a^2} = 1.651868$, and $\frac{c^2 - a^2}{a^2} = 0.651868$; and since this expression equals the sine of the angle PAM, it will be found from the tables that the angle $PAM = 40^\circ\ 41'$; and therefore twice this, or the angle MAN, will equal $81^\circ\ 22'$. Wherefore, if the extremities of the elastic ribbon be brought toward each other until they touch, they will form the curve $AMCNA$ [13]), [Fig. 8] and the two extremities will form at A an angle $= 81^\circ\ 22'$.

Sixth Class

30. If the two extremities A and B of the ribbon, after they have been brought together, should be drawn apart in opposite directions by an increased force, there will arise the curve of the shape $AMCNB$ (Fig. 9) which constitutes the sixth class.

Therefore in the curves belonging to this class, $\frac{c^2}{2a^2} > 0.825934$; but $\frac{c^2}{2a^2} < 1$. For if $c^2 = 2a^2$, there will arise the seventh class, to be explained in a moment. In these curves the angle PAM which the curve makes with the axis at A is greater than 40° -41′, but less than a right angle; for since its sine is $\frac{c^2 - a^2}{c^2}$, because $c^2 < 2a^2$, the sine is necessarily less than 1, and hence the angle PAM cannot become a right angle unless $c^2 = 2a^2$.

Seventh Class

31. Now let $c^2 = 2a^2$, in which case the seventh class is constituted, and the character of the curve will be expressed by the equation

$$dy = \frac{(a^2 - x^2)\, dx}{x\sqrt{2a^2 - x^2}},$$

from which it is gathered that the branches A and B of the curve (Fig. 10) are extended indefinitely, in such a way that the straight line AB becomes the asymptote of the curve. Therefore each branch AMC and BNC will become infinite, as is understood from the series discovered above for the arc AC; for

$$AC = \frac{\pi a}{2\sqrt{2}}\left(1 + \frac{1^2}{2^2} + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2} + \ldots\right),$$

the sum of which series is infinite. If therefore the length of the ribbon AC be finite and equal to f, it is necessary that $a = o$, and hence also $CD = c = o$; therefore the ribbon after it has been curved to a knot, in this case will be extended again in a straight line, for which an infinite force will be needed. But if the ribbon be infinitely long, it will form a knotted curve converging to the asymptote AB, CD being equal to c. Now the equation for this curve can be integrated by the aid of logarithms, for

$$y = \sqrt{c^2 - x^2} - \frac{c}{2} \log \frac{c + \sqrt{c^2 - x^2}}{x},$$

taking the abscissa x on the diameter DC in such a way that

$DQ = x$ and $QM = y$; for the ordinate y disappears when $x = CD = c$; also at the knot O the ordinate disappears. To find this point let us put

$$\frac{2\sqrt{c^2 - x^2}}{c} = \log \frac{c + \sqrt{c^2 - x^2}}{x}.$$

Let ϕ be the angle whose cosine is $\frac{x}{c}$ and whose sine is $\frac{\sqrt{c^2 - x^2}}{c}$, then $2 \sin \phi = \log \tan (45^\circ + \frac{\phi}{2})$.

The logarithm must be taken from a table of natural logarithms. If a table of this kind be lacking, let there be taken from a table of common logarithms the logarithm of the tangent of the angle $(45^\circ + \frac{\phi}{2})$, from the characteristic of which let 10 be subtracted, and let the remainder be ω; by so doing
$2 \sin \phi = 2.30258509\ \omega$ [14]).
Taking common logarithms once more, we have
$\log 2 + \log \sin \phi = \log \omega + 0.3622156886$,
or $\log \sin \phi = \log \omega + 0.0611856930$.
By trying this artifice, a value of the angle ϕ very close to the true value will soon be secured; whence by the rule of the false value the true value of the angle ϕ will be determined, and from it the abscissa $x = DO$. Now in this way the angle ϕ is found to be $73^\circ\ 14'\ 12''$, whence it results that $\frac{x}{c} = 0.2884191$, and $\frac{\sqrt{c^2 - x^2}}{c} = 0.9575042$. But the angle QOM is $2\phi - 90^\circ = 56^\circ\ 28'\ 24''$, and therefore the angle $MON = 112^\circ\ 56'\ 48''$. Therefore, since in the fifth class the angle at the knot was $81^\circ\ 22'$, in the sixth class the angle MON at the knot will be contained between the limits $81^\circ\ 22'$ and $112^\circ\ 56'\ 48''$. In the fourth class, if the knot appears, its angle will be less than $81^\circ\ 22'$.

Eighth Class

32. Now let $c^2 > 2a^2$, and $c^2 = 2a^2 + g^2$; since $a^2 = \frac{c^2 - g^2}{2}$,

the equation of the curve will be

$$dy = \frac{(x^2 - \frac{1}{2}c^2 - \frac{1}{2}g^2)\,dx}{\sqrt{(c^2 - x^2)(x^2 - g^2)}}.$$

By this equation the eighth class is expressed (FIG. 11), and if the straight line *dDd* represents the direction of the external force, then $x = DQ$, and $y = QM$. First therefore, it is clear that the ordinate y cannot be real unless $x > g$; but x cannot exceed the straight line $DC = c$, whence, putting $DF = g$, the whole curve will be contained between the straight lines parallel to *dd* drawn through the points *C* and *F*, and these lines will be tangent to the curve. Now it is indifferent which one of the straight lines c and g be the greater, provided only that they be unequal; for the equation is not changed if the straight lines c and g be interchanged. Furthermore this curve will also have an infinite number of diameters parallel to one another *DC*, *dc*, *dc*, etc., and also straight lines which are drawn through the individual points *G* and *H*, likewise normal to *dDd* [15]). But nowhere along the whole curve will there be a point of contraflexure, and therefore the continual curvature will progress in both directions indefinitely, as the figure shows; and the angles *MON*, *mon*, etc., which are made at the knots will be greater than 112° -56′ -48″.

NINTH CLASS

33. Since in the eighth class are contained not only the cases in which $g^2 < c^2$, but also those in which $g^2 > c^2$, there remains but the one case in which $c = g$; in which case, because $CF = o$, the curve is reduced, vanishing into space. But on the other hand, if we take c and g both as infinite, but in such a way that their difference is finite, the curve will occupy a finite space. Therefore, to find this curve, let $g = c - 2h$, and $x = c - h - t$, and, because $c = \infty$, but the quantities h and t are finite,

$$\frac{c^2}{2} + \frac{g^2}{2} = c^2 - 2ch, \text{ and } x^2 - \frac{c^2}{2} - \frac{g^2}{2} = -2ct;$$

Then $c^2 - x^2 = 2c(h + t)$, and $x^2 - g^2 = 2c(h - t)$, from which the following equation

$$dy = \frac{tdt}{\sqrt{h^2 - t^2}}$$

for the circle will result. Therefore the elastic band in this case will be curved into a circle, as we have already observed above. Wherefore the circle will constitute the ninth and last class.

34. Now that we have enumerated the classes, it will be easy, in any given case, to determine to which class the curve belongs. Let the elastic ribbon be fixed in the wall at G (FIG. 12), and from the end A let there be hung the weight P, by which the ribbon is curved into the shape GA. Let the tangent AT be drawn, and the whole decision will have to be sought from the angle TAP. For if this angle be acute, the curve will belong to the second class; but if it be a right angle, the curve will belong to the third class, and it will be an elastic rectangular curve. But if the angle TAP be obtuse, yet less than 130° -41′, the curve will belong to the fifth class; if, however, the angle TAP be greater than 130° -41′, the curve will belong to the sixth class. Now it would belong to the seventh class if the angle should be equal to two right angles, but that cannot happen. This class, therefore, together with the following classes, cannot be produced by hanging a weight directly to the ribbon.

35. Now in order that it may appear how the remaining classes can be produced by curving the ribbon, let a rigid rod AC be firmly fastened at A, the end of the ribbon fixed at B (FIG 3), and let the weight P, which draws in the direction CD, be appended at C. Let the interval AC be h, the absolute elasticity of the ribbon be Ek^2, and the sine of the angle MAP which the ribbon makes with the horizontal at A be m. All this being stipulated, if we let the abscissa $AP = t$, and the ordinate $PM = y$, the following equation will be found for the curve

$$\text{I. } dy = \frac{dt\,(mEk^2 - Pht - \frac{1}{2}Pt^2)}{\sqrt{E^2k^4 - (mEk^2 - Pht - \frac{1}{2}Pt^2)^2}}.$$

Now let $CP = x = h + t$, whereby the equation is reduced to the form which we have used in the division of the classes, viz.,

$$\text{II. } dy = \frac{dx\,(mEk^2 + \frac{1}{2}Ph^2 - \frac{1}{2}Px^2)}{\sqrt{E^2k^4 - (mEk^2 + \frac{1}{2}Ph^2 - \frac{1}{2}Px^2)^2}}$$

which, compared with the form

$$dy = \frac{dx\,(a^2 - c^2 + x^2)}{\sqrt{(c^2 - x^2)\,(2a^2 - c^2 + x^2)}},$$

or

$$\text{III. } dy = \frac{dx\,(a^2 - c^2 + x^2)}{\sqrt{a^4 - (a^2 - c^2 + x^2)^2}},$$ [16])

will give $\frac{1}{2}Pa^2 = Ek^2$, or $a^2 = \frac{2Ek^2}{P}$; and $\frac{1}{2}Pc^2 - \frac{1}{2}Pa^2 = mEk^2 + \frac{1}{2}Ph^2$; therefore

$$c^2 = \frac{2\,(1 + m)\,E\,k^2}{P} + h^2$$

36. Therefore the curve will belong to the second class if $\frac{2\,m\,E\,k^2}{P} + h^2 < o$, *or* $P < \frac{-\,2mEk^2}{h^2}$,

Hence unless the angle PAM be negative, the force P will have to be negative, and the rod at C will have to be drawn upwards. The curvature will belong to the third class if $P = \frac{-\,2mEk^2}{h^2}$. The fourth class will appear if $2mEk^2 + Ph^2 > o$, but at the same time $2mEk^2 + Ph^2 < 2aEk^2$, a being equal to 0.651868. But if $P = \frac{2(a - m)Ek^2}{h^2}$, then the curve will belong to the fifth class. If, however, $Ph^2 > 2(a - m)Ek^2$, but at the same time $Ph^2 < 2(1 - m)Ek^2$, the curve is to be referred to the sixth class. The seventh class will arise if $Ph^2 = 2(1 - m)Ek^2$. The eigth class will be obtained if $Ph^2 > 2(1 - m)Ek^2$. Wherefore if the angle PAM be a right angle, because $1 - m = o$, the curve will always belong to the eighth class. Finally, the ninth class will arise if $h = \infty$, as I have already observed above.

Strength of Columns

37. That which has been observed above about the first class can help us judge of the strength of columns. For let the column AB (Fig. 13), sustaining the load P, be placed vertically upon the base A. If the column be so constituted

that it cannot slip, nothing else need be feared from the weight P, if it be not excessively great, except the bending of the column; therefore in this case the column can be considered as elastic. Let the absolute elasticity of the column be Ek^2, and let its height $AB = 2f = a$. As we have seen above in section 25, the force necessary to bend this column even in the least degree is

$$\frac{\pi^2 Ek^2}{4f^2} = \frac{\pi^2}{a^2} \cdot Ek^2.$$

Therefore, unless the load P to be borne be greater than $\frac{E\pi^2k^2}{a^2}$, there will be absolutely no fear of bending; on the other hand, if the weight P be greater, the column will be unable to resist bending. Now when the elasticity of the column and likewise its thickness remain the same, the weight P which it can carry without danger will be inversely proportional to the square root of the height of the column; and a column twice as high will be able to bear only one-fourth of the load. This principle can, therefore, be applied in the case of wooden columns, since they are subject to bending.

Determination of the Absolute Elasticity by Experiment

38. Now in order that the force and the bending of any elastic ribbon whatsoever may be determined *a priori*, it is necessary that the absolute elasticity which we have hitherto expressed by Ek^2, be known. This can be conveniently determined by a single experiment. Let the uniform elastic ribbon, the absolute elasticity of which is to be investigated, be fixed at one end F, in a solid wall GK (Fig. 14), in such a way that it is held horizontally, for here we may neglect the weight of the ribbon. To the other end H let there be hung any weight P by which the ribbon is curved to the position AF. Let the length of the ribbon $AF = HF = f$, the horizontal distance $AG = g$, and the vertical distance $GF = h$, all of which values will be known by experiment. Now let this curve be compared with the general equation

$$dy = \frac{(c^2 - a^2 - x^2)\, dx}{\sqrt{(c^2 - x^2)\,(2a^2 - c^2 + x^2)}},$$

in which, if a and c have been determined by f, g, and h, the curving force P will equal $\frac{2Ek^2}{a^2}$. Therefore the absolute elasticity $Ek^2 = \frac{1}{2} Pa^2$.

39. Because now the tangent at F is horizontal, $\frac{dy}{dx} = o$, and therefore $x = \sqrt{c^2 - a^2}$. Hence $AG = g = \sqrt{c^2 - a^2}$, and $a^2 = c^2 - g^2$; and therefore

$$dy = \frac{(g^2 - x^2)\, dx}{\sqrt{(c^2 - x^2)\,(c^2 - 2g^2 + x^2)}},$$

and if we let $x = g$ here, y will have to be equal to $GF = h$, or $s = AF = f$; whence

$$ds = \frac{(c^2 - g^2)\, dx}{\sqrt{(c^2 - x^2)\,(c^2 - 2g^2 + x^2)}}$$

Now if the weight P be taken as very small, so that the ribbon be depressed only a very little, then c will become a very great quantity, and therefore

$$\frac{1}{\sqrt{(c^2 - \chi^2)\,(c^2 - 2g^2 + \chi^2)}} = (c^4 - 2c^2g^2 + 2g^2 - x^4)^{-\frac{1}{2}}$$

$$= \frac{1}{c^2} + \frac{g^2}{c^4} + \frac{3g^4}{2c^6} - \frac{g^2x^2}{c^6} - \frac{3g^4x^2}{8} + \frac{x^4}{2c^6} + \frac{3g^2x^4}{2c^8} + \ldots,$$

and therefore by integration we have approximately

$$s = \frac{(c^2 - g^2)x}{c^2} + \frac{(c^2 - g^2)g^2x}{c^4} + \frac{3(c^2 - g^2)g^4x}{2c^6} - \frac{(c^2 - g^2)g^2x^3}{3c^6} - \frac{(c^2 - g^2)g^4x^3}{c^8} + \frac{(c^2 - g^2)x^5}{10c^6} + \frac{3(c^2 - g^2)g^2x^5}{10c^8} + \ldots,$$

and

$$y = \frac{g^2x}{c^2} + \frac{g^4x}{c^4} + \frac{3g^6x}{2c^6} - \frac{g^4x^3}{3c^6} - \frac{g^6x^3}{c^8} + \frac{g^2x^5}{10c^6} + \frac{3g^4x^5}{10c^8} - \frac{x^3}{3c^2} - \frac{g^2x^3}{3c^4} - \frac{g^4x^3}{2c^6} + \frac{g^2x^5}{5c^6} + \frac{3g^4x^5}{5c^8} - \frac{x^7}{14c^6} - \frac{3g^2x^7}{14c^8} + \ldots$$

Now letting $x = g$, we have

$$f = g + \frac{4g^5}{15c^4},$$ [17]) and $h = \frac{2g^3}{3c^2} + \frac{2g^5}{3c^4}$.

Therefore if the straight line $FG = h$ be called into use

$$c^2 = \frac{2g^3}{3h}$$

and $$a^2 = \frac{g\,(2g^2 - 3gh)}{3h}$$

Whence the absolute elasticity is secured as

$$Ek^2 = \frac{Pg^2\ (2g - 3h)}{6h}$$

This value will differ hardly at all from the true value, provided that a not too great curvature of the ribbon be caused.

40. Now this absolute elasticity Ek^2 will depend in the first place upon the character of the material out of which the ribbon has been made; whence one material is said to have more elasticity than another. Also, in the second place, it so depends upon the breadth of the ribbon that the expression Ek^2 ought everywhere to be proportional to the breadth, other things being equal. But thirdly, the thickness of the ribbon contributes a great deal to determining the value of Ek^2, which seems to be composed in such a way that, other things being equal, Ek^2 is proportional to the square of the thickness. Therefore, taken all together, the expression Ek^2 will have a total ratio composed of the ratio of the elasticity of the material, the breadth of the ribbon, and the square of the thickness. Hence by experiments in which it is possible to measure the width and thickness, the elasticity of all materials can be compared one with another and determined.

Curvature of Elastic Ribbons of Variable Cross Section

41. As hitherto I have taken the absolute elasticity Ek^2 as constant throughout the whole length of the ribbon the curvature of which I have determined, so the solution can also be made by the same method if the quantity Ek^2 be taken as a variable in any manner. In other words, if the absolute elasticity of the ribbon AM (Fig. 2) be any function S of the arc $AM = s$, and if the radius of curvature at M be R, the curve AM, which

the ribbon takes on, will be so constituted that among all other curves of the same length, $\int \frac{Sds}{R^2}$ is a minimum. This case will therefore be solved by the second general formula [18]).

Let $dy = pdx$, $dp = qdx$, and $dS = Tds$. Among all curves in which $\int dx \sqrt{1 + p^2}$ is of the same magnitude, that curve is to be determined in which $\int \frac{Sq^2dx}{(1 + p_2)^{\frac{5}{2}}}$ is a minimum. The first formula, $\int dx \sqrt{1 + p^2}$, gives for a differential value $\frac{d}{dx} \frac{p}{\sqrt{1 + p^2}}$.

The second formula $\int \frac{Sq^2dx}{(1 + p^2)^{\frac{5}{2}}}$, compared with $\int Zdx$, will give $Z = \frac{Sq^2}{(1 + p^2)^{\frac{5}{2}}}$.

Now if we let $dZ = Ld\Pi + Mdx + Ndy + Pdp + Qdq$, where $\Pi = \int [Z]dx$, then $d[Z] = [M]dx + [N]dy + [P]dp$, and $Ld\Pi = \frac{q^2Tds}{(1 + p^2)^{\frac{5}{2}}}$,

whence $L = \frac{q^2T}{(1 + p^2)^{\frac{5}{2}}}$, and $d\Pi = ds = dx \sqrt{1 + p^2}$.

Therefore $[Z] = \sqrt{1 + p^2}$, $[M] = o$, $[N] = o$, and $[P] = \frac{p}{\sqrt{1 + p^2}}$.

But then $M = o$, $N = o$, and also $P = -\frac{5\,Sq^2p}{(1 + p^2)^{\frac{7}{2}}}$, and $Q = \frac{2Sq}{(1 + p^2)^{\frac{5}{2}}}$, so that $dZ = \frac{q^2dS}{(1 + p^2)^{\frac{5}{2}}} + Pdp + Qdq$.

42. Now let $\int Ldx = \int \frac{q^2Tdx}{(1 + p^2)^{\frac{5}{2}}} = \int \frac{q^2dS}{(1 + p^2)^3}$, and let H be the value of this integral when $x = a$. The consideration of the constant a will soon disappear again from the calculation. Therefore $V = H - \int \frac{q^2\,dS}{(1 + p^2)^3}$, whence

the differential expression will become $-\frac{dP}{dx} - \frac{d}{dx}[P]V + \frac{d^2Q}{dx^2}$.

Wherefore from these two differential values, the following equation for the curve sought will arise:

$$\alpha \frac{d}{dx} \frac{p}{\sqrt{1+p^2}} = + \frac{dP}{dx} + \frac{d}{dx}[P]\,V - \frac{d^2Q}{dx^2}.$$

This, being integrated, gives

$$\frac{\alpha p}{\sqrt{1+p^2}} + \beta = P + [P]\,V - \frac{dQ}{dx}.$$

or

$$\frac{\alpha p}{\sqrt{1+p^2}} + \beta = \frac{Hp}{\sqrt{1+p^2}} - \frac{p}{\sqrt{1+p^2}}\int \frac{q^2\,dS}{(1+p^2)^3} + P - \frac{dQ}{dx}.$$

The constant H can be absorbed into the arbitrary constant α, whereby the constant a disappears from the calculation. On this account, the following equation will result:

$$\frac{\alpha p}{\sqrt{1+p^2}} + \beta = P - \frac{dQ}{dx} - \frac{p}{\sqrt{1+p^2}}\int \frac{q^2\,dS}{(1+p^2)^3}.$$

43. Let this equation be multiplied by $dp = qdx$, and there will result

$$\frac{\alpha\,pdp}{\sqrt{1+p^2}} + \beta\,dp = Pdp - qdQ - \frac{pdp}{\sqrt{1+p2}}\int \frac{q^2dS}{(1+p^2)^3}.$$

Now since $dZ = \frac{q^2\,dS}{(1+p^2)^{\frac{3}{2}}} + Pdp + Qdq$,

then $Pdp = dZ - Qdq - \frac{q^2\,dS}{(1+p^2)^{\frac{3}{2}}}$.

If this value is substituted, the following integrable equation will result:

$$\frac{\alpha\,pdp}{\sqrt{1+p^2}} + \beta\,dp = dZ - qdQ - Qdq - \frac{q^2\,dS}{(1+p^2)^{\frac{3}{2}}} - \frac{pdp}{\sqrt{1+p^2}}\int \frac{q^2\,dS}{(1+p^2)^3},$$

the integral of which is

$$\alpha\sqrt{1+p^2} + \beta p + \gamma = Z - Qq - \sqrt{1+p^2}\int \frac{q^2\,dS}{(1+p^2)^3},$$

$$\text{or } \alpha\sqrt{1+p^2}+\beta p+\gamma=-\frac{Sq^2}{(1+p^2)^{\frac{3}{2}}}-\sqrt{1+p^2}\int\frac{q^2dS}{(1+p^2)^3}.$$

In order to eliminate the integral sign, divide the equation by $\sqrt{1+p^2}$ and differentiate again, obtaining

$$\frac{\beta\,dp}{(1+p^2)^{\frac{3}{2}}}-\frac{\gamma\,pdp}{(1+p^2)^{\frac{3}{2}}}+\frac{2\,q^2dS}{(1+p^2)^3}+\frac{2\,Sqdq}{(1+p^2)^3}-\frac{6\,Spq^2dp}{(1+p^2)^4}=0,$$

which, multiplied by $\frac{(1+p^2)^{\frac{3}{2}}}{2q}$, gives

$$\frac{\beta\,dp}{2q}-\frac{\gamma\,pdp}{2q}+\frac{qdS+Sdq}{(1+p^2)^{\frac{3}{2}}}-\frac{3\,Spqdp}{(1+p^2)^{\frac{5}{2}}}=0.$$

The integral of this, because $dp=qdx$, and $dy=pdx$, will be

$$\alpha+\tfrac{1}{2}\beta x-\tfrac{1}{2}\gamma y+\frac{Sq}{(1+p^2)^{\frac{3}{2}}}=0.$$

But $-\frac{(1+p^2)^{\frac{3}{2}}}{q}$ is the radius of curvature R; whence, by doubling the constants β and γ, the following equation will arise:

$$\frac{S}{R}=\alpha+\beta x-\gamma y.$$

This equation agrees admirably with that which the second or direct method supplies. For let $\alpha+\beta x-\gamma y$ express the moment of the bending power, taking any line you please as an axis [19]), to which moment the absolute elasticity S, divided by the radius of curvature R must be absolutely equal. Thus, therefore, not only has the character of the elastic curve observed by the celebrated BERNOULLI been most abundantly demonstrated, but also the very great utility of my somewhat difficult formulas has been established in this example.

44. If, therefore, the curve be given which a variable elastic ribbon, acted upon by the force $CD=P$ (FIG. 3) forms, the absolute elasticity of the ribbon at any point can be found. For, taking the straight line CP, which is normal to the direction of the force, as an axis, and putting $CP=x$, $PM=y$, the arc of the curve $AM=s$, and the radius of curvature at M equal to R, because the moment of the force with reference to the

point M is Px, then $\frac{S}{R} = Px$, and therefore the absolute elasticity S, at the point M, is PRx. Hence, since the radius of curvature R is known at every point when the curve is given, the absolute elasticity at any point becomes known. Therefore, if the material of the ribbon together with its thickness be everywhere the same, but the width is variable, because the absolute elasticity is proportional to the width, the width of the ribbon at every point is learned from the form of the curve.

45. Let the triangular tonguelet fAf (FIG. 15) be cut out of an elastic ribbon of uniform thickness. Since the width mm at any section M is proportional to the length AM, if we let $AM = s$, the absolute elasticity at M will be proportional to s. Let the absolute elasticity be Eks, and to the ribbon fastened at the end ff horizontally in a wall let there be hung, at the point A, the weight P, by which the median straight line AF is bent into the curve FmA (FIG. 14), the character of which curve is sought. Now on the horizontal axis let the abscissa $Ap = x$, the ordinate $pm = y$, and the arc $Am = s$; then $Px = \frac{Eks}{R}$, in which R denotes the radius of curvature at m. Let this equation be multiplied by dx, and because $R = -\frac{ds^3}{dxd^2y}$, assuming dx as constant, we have $Px.dx = \frac{-Eksdx^2\,d^2y}{ds^3}$

or $$\frac{Pxdx}{Ek} + \frac{sdx^2.d^2y}{ds^3} = 0$$

But since $\frac{d}{ds}\,sdy = \frac{sd^2y}{ds} - \frac{sdyd^2s}{ds^3} + dy = \frac{sdx^2.d^2y}{ds^3} + dy$,

and because $d^2s = \frac{dyd^2y}{ds}$, then

$$\int \frac{sdx^2d^2y}{ds^3} = \frac{sdy}{ds} - y.$$

Whence by integration,

$$\frac{Px^2}{2Ek} + a = \frac{-sdy}{ds} + y.$$

46. Let $dy = pdx$, so that $ds = dx\sqrt{1+p^2}$, and $\frac{2Ek}{P} = c$;

then $a + \frac{x^2}{c} = y - \frac{sp}{\sqrt{1+p^2}}$, and therefore

$$\frac{a\sqrt{1+p^2}}{p} + \frac{x^2\sqrt{1+p^2}}{cp} = \frac{y\sqrt{1+p^2}}{p} - s.$$

This, differentiated, gives

$$\frac{-adp}{p^2\sqrt{1+p^2}} + \frac{2\,x\,dx\sqrt{1+p^2}}{cp} - \frac{x^2\,dp}{cp^2\sqrt{1+p^2}} = \frac{dy\sqrt{1+p^2}}{p}$$

$$-y\frac{dp}{p^2\sqrt{1+p^2}} - dx\sqrt{1+p^2} = \frac{-y\,dp}{p^2\sqrt{1+p^2}}.$$

Hence $a - y = \frac{2pxdx\,(1+p^2)}{cdp} - \frac{x^2}{c}$ results.

Let dp be taken as constant and differentiate; then

$$-pdx = \frac{2\,pxd^2x\,(1+p^2)}{cdp} + \frac{2\,pdx^2\,(1+p^2)}{cdp} + \frac{2\,xdx\,(1+3p^2)}{c} - \frac{2xdx}{c},$$

or

$$cdxdp + 2xd^2x\,(1+p^2) + 2dx^2\,(1+p^2) + 6pxdx = o.$$

A further solution of this equation is impossible. The most simple equation for the curve is the following:

$$\frac{yds - sdy}{ds} = \frac{Px^2}{2Ek},$$

for when $x = o$, both y and s must vanish, and the constant $a = o$.

The Curvature of Elastic Ribbons which in Their Natural State are Not Straight

47. In the previous discussion the curvature of a ribbon, whether uniformly elastic or not, is determined if it be subjected to a single force, and, which is especially to be noted, if the ribbon be naturally straight. But if the ribbon in its natural state be already curved, then it will certainly take on a different curvature due to the acting force. To find this, one must know its natural shape in addition to its elasticity and the acting force. Let,

therefore, the elastic ribbon *Bma* (FIG. 16) be naturally curved; let the elasticity of it be everywhere the same, viz. Ek^2, and let it be curved by the force P into the shape *BMA*. Through the point A let there be drawn the straight line *CAP* normal to the direction of the acting force, and let this line be taken as the axis; let also the distance $AC = c$, the abscissa $AP = x$, the ordinate $PM = y$; and the moment of the acting force at the point M be equal to $P(c + x)$.

48. Furthermore let the radius of curvature of the curve sought be equal to R at the point M; let the arc *am* in the natural state be $AM = s$, and let the radius of curvature at the point m be r; this radius, because the curve *amB* is known, will be given by the arc s. At M, therefore, because the curvature is greater, the radius of curvature R is less than r, and the excess of the elementary angle over the angle in the natural state will be $\frac{ds}{R} - \frac{ds}{r}$, which excess will be the effect produced by the acting force. Wherefore $p(c + x) = Ek^2 \left(\frac{1}{R} - \frac{1}{r}\right)$, which, since r is given by s, will be the equation of the curve sought; and this considered thus cannot be reduced to one of the previously described classes.

49. Therefore let us assume that the ribbon has a circular shape *amB* in its natural state; r will be the radius a of that circle, whence $P(c + x) = Ek^2 \left(\frac{1}{R} - \frac{1}{a}\right)$. Let this equation be multiplied by dx and integrated; then [see sec. 5 towards the end]

$$\frac{P}{Ek^2}\left(\frac{x^2}{2} + cx + f\right) = \frac{-dy}{ds} - \frac{x}{a},$$

will arise, which, if $c - \frac{Ek^2}{Pa}$ be written for c, will go over into

$$\frac{P}{Ek^2}\left(\frac{x^2}{2} + cx + f\right) = -\frac{dy}{ds}.$$

This is the same equation that we discovered above for the ribbon

which was straight in its natural condition [sec. 5]. Let, therefore, the ribbon which is circular in its natural condition be curved into the same curves which are produced for the ribbon that is straight in its natural state; of course, the place of the application of the force, or the distance $AC = c$ will have to vary for each case according to the given law. Therefore the same nine classes of curves will appear for the figures which the ribbon that is circular in its natural state can produce, and these we have enumerated above. For the circular ribbon, if the distance AC be taken as infinite, can be drawn first into a straight line [see class 9]; then any force whatever applied in addition will produce the same effect as if it were applied alone to the elastic ribbon which was straight in its natural state.

50. Now let us assume that, whatever be the natural shape of the ribbon, the point C is infinitely distant, in such a way that the moment of the acting force be everywhere the same, and let the moment, when divided by Ek^2 be taken as $\frac{1}{b}$, then

$$\frac{1}{b} = \frac{1}{R} - \frac{1}{r} \text{ and } \frac{1}{R} = \frac{1}{b} + \frac{1}{r}.$$

Hence $\int \frac{ds}{R} = \frac{s}{b} + \int \frac{ds}{r}$

is the amplitude of the arc AM, just as $\int \frac{ds}{r}$ expresses the amplitude of the arc am, precisely as the celebrated JOHN BERNOULLI is accustomed to use the term amplitude in his superb treatise *De motu reptorio* [20].) Let therefore $\frac{s}{b} + \int \frac{ds}{r}$ be the arc in the circle whose radius equals 1, which, because r is given by s, will also be a known function of s. Hence the rectangular co-ordinates x and y will be found in such a way that

$$x = \int ds \sin\left(\frac{s}{b} + \int \frac{ds}{r}\right), \text{ and } y = \int ds \cos\left(\frac{s}{b} + \int \frac{ds}{r}\right),$$

whence the curve sought can be constructed by quadratures.

51. Hence the figure amB (FIG. 17), which the ribbon must

have in its natural state, can be determined, so that by the force P, acting in the direction AP, it can be unfolded into the straight line AMB. For letting $AM = s$, the moment of the force acting at the point M will equal Ps, and the radius of curvature at M will be infinite by hypothesis, or $\frac{1}{R} = o$. Now the arc am in its natural state being equal to s, and the radius of curvature at m being taken as r, because this curve is convex to the axis AB, the quantity r must be made negative. Hence $Ps = \frac{Ek^2}{r}$, or $rs = a^2$, which is the equation of the curve amB.

52. Therefore, since $\frac{1}{r} = \frac{s}{a^2}$, then $\int \frac{ds}{r} = \frac{s^2}{2a^2}$; or the amplitude of the arc am will vary as the square of the arc itself. Hence the rectangular co-ordinates x and y for the curve amB will be so defined that $x = \int ds \sin \frac{s^2}{2a^2}$, and $y = \int ds \cos \frac{s^2}{2a^2}$. In other words, in a circle whose radius is 1, the arc $\frac{s^2}{2a^2}$ will have to be cut off, the sine and cosine of which must be taken to determine the co-ordinates. Now from the fact that the radius of curvature constantly decreases the greater the arc $am = s$ is taken, it is manifest that the curve cannot become infinite, even if the arc s be infinite. Therefore the curve will belong to the class of spirals, in such a way that after an infinite number of windings it will roll up at a certain definite point as a center, which point seems very difficult to find from this construction. Analysis therefore must be considered to gain no slight advantage if anyone should discover a method by the aid of which at least an approximate value can be assigned for the integrals $\int ds \sin \frac{s^2}{2a^2}$, and $\int ds \cos \frac{s^2}{2a^2}$, in the case where s is taken as infinite. This seems to be a not unworthy problem upon which mathematicians may exercise their powers [21]).

53. Let $2a^2 = b^2$, and since

$$\sin \frac{s^2}{b^2} = \frac{s^2}{b^2} - \frac{s^6}{3!\,b^6} + \frac{s^{10}}{5!\,b^{10}} - \frac{s^{14}}{7!\,b^{14}} + \ldots\ldots,$$

$$\text{and } \cos \frac{s^2}{b^2} = 1 - \frac{s^4}{2!\,b^4} + \frac{s^8}{4!\,b^8} - \frac{s^{12}}{6!\,b^{12}} + \ldots\ldots,$$

the co-ordinates x and y of the curve sought can be conveniently expressed by infinite series; for

$$x = \frac{s^3}{3\,b^2} - \frac{s^7}{3!\,7\,b^6} + \frac{s^{11}}{5!\,11\,b^{10}} - \frac{s^{15}}{7!\,15\,b^{14}} + \ldots\ldots,$$

$$\text{and } y = s - \frac{s^5}{2!\,5\,b^4} + \frac{s^9}{4!\,9\,b^8} - \frac{s^{13}}{6!\,13\,b^{12}} + \ldots\ldots,$$

from which rapidly converging series, unless the arc s be assumed to be very great, the approximate values of the co-ordinates x and y can be determined sufficiently closely. But what values x and y acquire if the arc s be taken as infinitely great, can in no way be determined from these series.

54. Therefore, since putting $s = \infty$ makes a very great difficulty, aid can be brought to the inconvenience by the following method. Let $\frac{s^2}{b^2} = v$, then $s = b\sqrt{v}$, and $ds = \frac{bdv}{2\sqrt{v}}$ whence $x = \frac{b}{2}\int \frac{dv}{\sqrt{v}} \sin v$, and $y = \frac{b}{2}\int \frac{dv}{\sqrt{v}} \cos v$. And now I declare that the values for x and y when $s = \infty$ will be discovered by the following integral formulas:

$$x = \frac{b}{2}\int dv \left(\frac{1}{\sqrt{v}} - \frac{1}{\sqrt{\pi + v}} + \frac{1}{\sqrt{2\pi + v}} - \frac{1}{\sqrt{3\pi + v}} + \ldots\ldots\right) \sin v,$$

$$\text{and } y = \frac{b}{2}\int dv \left(\frac{1}{\sqrt{v}} - \frac{1}{\sqrt{\pi + v}} + \frac{1}{\sqrt{2\pi + v}} - \frac{1}{\sqrt{3\pi + v}} + \ldots\ldots\right) \cos v$$ [22])

if after integration v be taken as equal to π, where π denotes an

angle equal to two right angles. In this way, therefore, the placing of $s = \infty$ is indeed avoided ; but on the other hand, the infinite series

$$\frac{1}{\sqrt{v}} - \frac{1}{\sqrt{\pi + v}} + \frac{1}{\sqrt{2\pi + v}} - \frac{1}{\sqrt{3\pi + v}} + \ldots.$$

is introduced into the calculation, and since the sum of this series is as yet unknown, the resolution of the knot is still subject to a great difficulty.

The Curvature of an Elastic Ribbon at Individual Points Under the Action of Any Forces Whatsoever

55. It will be convenient also to study the curvature produced in an elastic ribbon by several forces, or indeed by an infinite number of forces, by the same method already given for studying the curvature of any elastic ribbon whatsoever if it be acted upon by a single force at a given point. But since it is not yet established just what expression in these cases is going to be either a maximum or a minimum, I shall use merely the direct method, in order that from the solution itself it may perchance be possible to discover that property which is either a maximum or a minimum. Therefore let the elastic ribbon that is straight in its natural state be brought into the position *AmM* (Fig. 18), first by the finite forces *P* and *Q* acting in the directions *CE* and *CF* normal to each other, and then by the infinitely small forces applied to the single elements $m\mu$ of the ribbon, and acting in the directions *mp* and *mq* parallel to *CE* and *CF*; all this being stipulated, the character of the curve produced in the ribbon *AmM* is required.

56. Let the straight line *FCA* produced be taken as an axis, and let $AC = c$, the abscissa $AP = x$, the ordinate $PM = y$, the arc of the curve $AM = s$, and the radius of curvature at *M* be *R*; let the absolute constant of the elasticity of the ribbon be Ek^2; and the sum of the moments arising from all the acting moments with respect to the point *M* must be equal to $\frac{Ek^2}{R}$.

Now in the first place, from the finite force P acting in the direction CE there arises the moment $P(c + x)$, acting in that direction in which the elastic forces are equilibrated. The moment Qy, arising from the other force Q, tends in the opposite direction, from which, due to the finite forces P and Q taken together, there arises the moment $P(c + x) - Qy$. Now let there be considered any intermediate element $m\mu$, and let its corresponding abscissa $Ap = \zeta$, and the ordinate $pm = \eta$; let the force acting upon the element $m\mu$ in the direction mp be dp, and the force acting in the direction mq be dq; then the moment of these forces about M will be $(x - \zeta)\, dp - (y - \eta)\, dq$.

57. Therefore to find the sum of all the moments, the point M, and consequently x and y, must be, for the time being, considered as constants, while only the co-ordinates ζ and η with the forces dp and dq are regarded as variable. Therefore the sum of the moments arising from the forces acting upon the arc Am will equal

$$xp - \int \zeta dp - yp + \int \eta dq,$$

where p expresses the sum of all the forces acting upon the arc AM applied in the direction parallel to pm, and q expresses the sum of all the forces acting upon the arc AM applied in the direction parallel to Ap. But $\int \zeta dp = \zeta p - \int p d\zeta$,

and $$\int \eta dq = \eta q - \int q d\eta,$$

whence the sum of the moments arising from the forces applied to the arc AM will be $(x - \zeta)p + \int p d\zeta - (y - \eta)q - \int q d\eta$. Now let the point m move to M; then $\zeta = x$, $\eta = y$, $d\zeta = dx$, and $d\eta = dy$; whence the sum of all the moments taken throughout the whole length of the arc AM will equal $\int p dx - \int q dy$. Wherefore, for the curve sought, the following equation will be obtained :

$$\frac{Ek^2}{R} = P(c + x) - Qy + \int p dx - \int q dy.$$

Here p expresses the sum of all the vertical forces, or those acting in the direction of the ordinates MP, and q expresses the sum of all the horizontal forces, or those acting in the direction of MQ parallel to AP, throughout the whole arc AM.

58. If the expressions $\int p dx$ and $\int q dy$ cannot be integrated, the equations found by differentiation will have to be freed from these integral expressions, whence the following equation will be had :

$$\frac{-Ek^2 dR}{R^2} = Pdx - Qdy + pdx - qdy.$$

But if neither p nor q can be expressed in a finite number of terms, inasmuch as they already express the sums of an infinite number of forces infinitely small, then by a further differentiation, the finite values p and q will have to be eliminated, so that there remain only dp and dq, with the differentials of the second order d^2p and d^2q.

Now there will arise, after the first differentiation,

$$-Ek^2 d\frac{dR}{R^2 dx} = dp - (Q + q)\, d\frac{dy}{dx} - \frac{dy}{dx}\, dq.$$

Let $\frac{dy}{dx} = \omega$, and, when the equation has been differentiated again, we get

$$-Ek^2 d\frac{\frac{dR}{R^2 dx}}{d\omega} = d\frac{dp}{d\omega} - 2dq - \omega\, d\frac{dq}{d\omega}.$$

This equation contains differentials of the fourth order.

59. In place of the vertical and horizontal forces p and q, let two forces be applied to the ribbon—the one normal, $MN = dv$, and the other tangential, $MT = dt$ [FIG. 18]. Hence

$$dp = \frac{dx\, dv}{ds} + \frac{dy\, dt}{ds}, \text{ and } dq = \frac{dx\, dt}{ds} - \frac{dy\, dv}{ds},$$

and because $dy = \omega dx$, and $ds = dx\sqrt{1 + \omega^2}$,

$$\text{then } dp = \frac{dv}{\sqrt{1+\omega^2}} + \frac{\omega\, dt}{\sqrt{1+\omega^2}}, \text{ and } dq = \frac{dt}{\sqrt{1+\omega^2}} - \frac{\omega\, dv}{\sqrt{1+\omega^2}}.$$

When these values are substituted in the last equation of the preceding paragraph, the following equation will result :

$$-Ek^2 d\frac{\frac{dR}{R^2 dx}}{dv} = \frac{-dt}{\sqrt{1+\omega^2}} + \frac{2\omega\, dv}{\sqrt{1+\omega^2}} + \sqrt{1+\omega^2}\, d\frac{dv}{d\omega}.$$

This equation becomes integrable when multiplied by $\sqrt{1+\omega^2}$. For the sake of brevity, let $z = \frac{dR}{R^2 dx}$; then

$$A - t + \frac{dv\,(1+\omega^2)}{d\omega} = -Ek^2\left[\frac{dz\,\sqrt{1+\omega^2}}{d\omega} - \frac{\omega z}{\sqrt{1+\omega^2}} + \frac{1}{2R^2}\right].$$

$$= -Ek^2\left[\frac{1+\omega^2}{d\omega}\, d\,\frac{dR}{R^2\, dx\,\sqrt{1+\omega^2}} + \frac{1}{2R^2}\right].$$

But since $R = -\frac{(1+\omega^2)^{\frac{3}{2}}}{d\omega}dx$, then $d\omega = \frac{-(1+\omega^2)^{\frac{3}{2}}}{R}dx$, and by substituting the value of $d\omega$ we shall have, because $dx\sqrt{1+\omega^2} = ds$,

$$A - t - \frac{Rdv}{ds} = -Ek^2\left[\frac{1}{2R^2} - \frac{R}{ds}\, d\,\frac{dR}{R^2\, ds}\right]$$

Therefore by transposing, the following equation will arise:

$$t + \frac{R\,dv}{ds} - A = Ek^2\left[\frac{1}{2R^2} - \frac{R}{ds}\, d\,\frac{dR}{R^2\, ds}\right].$$

60. Now in the first place, it is clear that if the elastic force Ek^2 should vanish, the ribbon would be transformed into a perfectly flexible filament; and hence all the curves which a perfectly flexible filament can form when acted upon by any forces whatsoever are included in these equations. Thus if a filament be merely drawn downwards by its own weight, then $q = o$, and p will express the weight of the string AM, and therefore, by the first equation of section 58, $p\frac{dx}{dy} = Q =$ a constant, and $P = o$, which is the general equation for catenary curves of every kind. Now if a perfectly flexible filament be acted upon at various points by forces, the directions of which [FIG. 18] are normal to the curve itself, in such a way that, at the point M, the filament be drawn in the direction MN by a force dv, then, because $t = o$, it follows that $\frac{Rdv}{ds} = A$, a constant.

This is the general property of trough-shaped curves, and of all curves in which acting forces of this kind appear.

On the Curvature Produced in an Elastic Ribbon by its Own Weight

61. I return now to elastic ribbons about which there is offered the following investigation, which is especially worthy of note, viz., the kind of a figure an elastic ribbon takes on when curved by its own weight. Let AmM [Fig. 18] be the curve which is sought, and because only vertical forces due to gravity are acting upon it, $P = o$, $Q = o$, $q = o$, and p will express the weight of the ribbon AM, wherefore, if F be the weight of a ribbon of length a, because the ribbon is assumed to be uniform, $p = \frac{Fs}{a}$; whence the character of the curve will be expressed by the following equation [from sec. 58]:

$$\frac{-Ek^2dR}{R^2} = \frac{Fs\,dx}{a}$$

Let the amplitude of the curve be $\int \frac{ds}{R} = u$; then $R = \frac{ds}{du}$, and $dx = ds \sin u$; whence, assuming the element ds as a constant, the following equation will be found:

$$s\,ds \sin u - \frac{Eak^2}{F} \cdot \frac{d^2u}{ds} = o$$

which, as far as appears at first glance, cannot be reduced further.

62. Now especially worthy of note is the curve which a fluid of considerable depth produces in an elastic ribbon (Fig. 19). Let AMB be the figure sought, and letting $AP = x$, $PM = y$, and $AM = s$, the element Mm will be drawn in the normal direction MN by a force proportional to ds; whence $dv = nds$, and $dt = o$. Hence the vertical force $dp = ndx$, and the horizontal force $dq = -ndy$; whence $p = nx$, and $q = -ny$; and therefore the equation [of section 57] becomes

$$\frac{Ek^2}{R} = P(c + x) - Qy + \tfrac{1}{2}nx^2 + \tfrac{1}{2}ny^2.$$

The co-ordinates x and y can be increased or diminished by constant quantities in such a way that the equation for the curve takes on the following form :

$$x^2 + y^2 = A + \frac{B}{R}.$$

Now if this equation be multiplied by $xdx + ydy$, and if we put $dy = \omega dx$, it becomes integrable, for

$$\int \frac{xdx + ydy}{R} = -\int \frac{x + y\omega}{(1 + \omega^2)^{\frac{1}{2}}}\, d\omega = \frac{y - \omega x}{\sqrt{1+\omega^2}} = \frac{ydx - xdy}{ds}.$$

Because of this, by changing the constant after integration, we shall have

$$(x^2 + y^2)^2 = A\,(x^2 + y^2) + B\,\frac{(ydx - xdy)}{ds} + C.$$

Let $\sqrt{x^2 + y^2} = z$, and $y = uz$; whence $x = z\sqrt{1 - u^2}$; therefore $ydx - xdy = -\dfrac{z^2 du}{\sqrt{1 - u^2}}$, and $ds = \sqrt{dz^2 + \dfrac{z^2 du^2}{1-u^2}}$.

Therefore by placing

$$\frac{du}{\sqrt{1 - u^2}} = dr, \text{ then } z^4 - Az^2 - C = -\frac{Bz^2 dr}{\sqrt{dz^2 + z^2\, dr^2}},$$

and hence

$$dr = \frac{du}{\sqrt{1 - u^2}} = \frac{dz\,(z^4 - A\,z^2 - C)}{z\,\sqrt{B^2\,z^2 - (z^4 - A\,z^2 - C)^2}}.$$

Therefore this curve, if $A = o$ and $C = o$, will be algebraic, for we shall have the following equation

$$dr = \frac{du}{\sqrt{1 - u^2}} = \frac{z^2\, dz}{\sqrt{B^2 - z^6}} = \frac{3z^2\, dz}{3\sqrt{a^6 \quad z^6}},$$

which, being integrated, gives

$$\text{arc sin u} = \frac{1}{3} \text{ arc sin } \frac{z^3}{a^3},$$

$$\text{or } \frac{z^3}{a^3} = 3u - 4u^3 = \frac{3y}{z} - \frac{4y^3}{z^3},$$

whence $z^6 = 3a^3\, yz^2 - 4a^3\, y^3$.
Or, since $z^2 = x^2 + y^2$, $(x^2 + y^2)^3 = 3a^3x^2y - a^3y^3$, or
$x^6 + 3x^4y^2 + 3x^2y^4 + y^6 = 3a^3x^2y - a^3y^3$.

On the Oscillating Motions of Elastic Ribbons

63. Now from all this the oscillating motion of elastic curves brought into motion in any manner whatsoever can be determined. The illustrious Daniel Bernoulli first began to investigate this assuredly most important topic, and some years ago sent me the problem of determining the oscillations of an elastic ribbon fastened at one end in a solid wall, the solution of which I have published in "Commentarii Petropolitani" Vol. VII, (1740). Since that time, not only has it been my good fortune to treat the problem in a more convenient fashion, but also, through consultation with the celebrated Bernoulli, a number of other questions and considerations have been added, the elucidation of which, because of the relation of the subject matter, I shall here add. Now when the vibratory motion is sufficiently rapid, a musical tone is given by the vibrating ribbon, the pitch of which, and its relation to other tones, will be determined by these principles, with the aid of the theory of tones. And since the character of tones is very readily subject to experiment, by that fact the agreement of calculation with truth can be investigated, and the theory can be confirmed. In this fashion our knowledge of the nature of elastic bodies will be enlarged in no small measure.

64. Now it must first be noted that here our study is directed only to very small oscillations; and the interval through which the ribbon passes in oscillating is, as it were, infinitely small. But the utility and the application is not at all diminished by this limitation; for not only would oscillations be deprived of isochronism if they should take place through large spaces, but more than that, the formation of distinct tones, and that is what we are here primarily considering, requires very small oscillations. I therefore consider here, in the first place, a uniform elastic ribbon, naturally straight, one end of which is firmly fixed at B (Fig. 20) in an immovable pavement, in such a way that the ribbon, when left to itself, has the upright position BA. Let the length of this ribbon be $AB = a$, and its absolute elasticity at each point be Ek^2; its true weight we either neglect, or else

we cause it to be fixed in such a fashion that its position cannot be disturbed by gravity.

On the Oscillations of an Elastic Ribbon Fixed at One End in a Wall

65. Now this ribbon, acted on by any force whatsoever, performs very short vibrations passing through very small intervals *Aa* on either side of its natural position *BA*. Let *BMa* be any position whatsoever which the ribbon occupies while oscillating. Since this is only an infinitely short distance from its natural position *BPA*, the straight lines *MP* and *Aa* will at the same time represent the paths which the points *M* and *a* traverse, or rather these straight lines, when compared to the true paths, will differ from them by an infinitely small amount. Now to determine the oscillatory motion, it is absolutely necessary to know the character of the curve *BMa* which the ribbon takes on during oscillation. Therefore let $AP = x$, $PM = y$, the arc $aM = s$, the radius of curvature at M be R, and the very small interval $Aa = b$; also, from the conditions mentioned, the arc s will be approximately equal to the abscissa x, and accordingly dx can be taken for ds; for in comparison with dx, dy will be on the point of vanishing. And since, by assuming dx as constant, the general expression for the radius of curvature $\frac{ds^3}{dxd^2y}$ is, in the present case $R = \frac{dx^2}{d^2y}$, for the curve *BMa* turns its convex side to the axis *BA*; and because the ribbon has been firmly fixed in a wall at *B*, the straight line *AM* will be tangent to the curve at *B*.

66. All this being stipulated, in order to determine not only the character of the curve *BMa*, but also its oscillatory motion, let f be the length of a simple isochronous pendulum; for not only the nature of the case, but also the calculations to be instituted will show that the very small oscillations are isochronous. The acceleration by which the point M of the ribbon is drawn toward P will be $\frac{PM}{f} = \frac{y}{f}$. Wherefore, if the mass of the whole

ribbon be taken as M, which is expressed by the weight, the mass of the element $Mm = ds = dx$ is $\frac{Mdx}{a}$; whence the moving force drawing the element Mm in the direction MP is $\frac{Mydx}{af}$; and thus the forces by which the individual particles are actually brought into motion will be known, not only from the curve BMa, but also from the length f of the simple isochronous pendulum. But, since the ribbon is, as a matter of fact, incited to motion by the elastic force, when this is known, and the nature of the curve is known, the length of the simple isochronous pendulum will also be determined by them.

67. Therefore, since the ribbon is moved exactly as if there had been applied to each element Mm of it, in the direction MP, forces equaling $\frac{Mydx}{af}$, it follows that, if to the single elements Mm of the ribbon, equal forces $\frac{Mydx}{af}$ should be applied in the opposite direction $M\pi$, the ribbon in the position BMa would be in a state of equilibrium. Hence the ribbon while oscillating will undergo the same curvature which it would take on when at rest, if at the individual points M it should be acted upon by the forces $\frac{Mydx}{af}$ in the direction $M\pi$.

Therefore by the rule discovered above in section 56 [and 57], let all these forces applied throughout the arc aM be collected, and there will appear the sum $\frac{M}{af}\int ydx$, which must be substituted in the place of p. Wherefore, since the remaining forces P, Q, and q which appeared there [sec. 56] are on the point of vanishing, the character of the curve will be expressed by the equation $\frac{Ek^2}{R} = \int pdx$,

whence we shall secure $\frac{Ek^2}{R} = \frac{M}{af}\int dx \int ydx$.

But since $R = \frac{dx^2}{d^2y}$, then $\frac{Ek^2\ d^2y}{dx^2} = \frac{M}{af}\int dx \int ydx$.

Differentiating, $\frac{Ek^2\ d^3y}{dx^2} = \frac{M}{af}\ dx \int ydx$,

and by differentiating again, the following differential equation of the fourth order will appear : $Ek^2\ d^4y = \frac{My\ dx^4}{af}$.

68. By this equation, therefore, the character of the curves *BMa* is expressed, and from that, if it be adapted to the case presented, the length f will be determined. That being known, the oscillatory motion itself will become known. But first the equation must be integrated, and since it belongs to the class of differential equations of the higher orders, the general integration of which I have shown in Vol. VII of the " Miscell. Berol.," the following integral equation will be found by substituting, for the sake of brevity, c^4 for $\frac{Ek^2af}{M}$:

$y = Ae^{\frac{x}{c}} + Be^{\frac{-x}{c}} + C\sin\frac{x}{c} + D\cos\frac{x}{c}$, where e denotes a number the hyperbolic logarithm of which is 1, and $\sin\frac{x}{c}$, and $\cos\frac{x}{c}$ are assumed to denote the sine and cosine of the arc $\frac{x}{c}$ in a circle, the radius of which is 1. But then A, B, C and D are four arbitrary constants introduced by four integrations, which must be determined by adapting the calculus to the present case.

69. Now the determination of the constants will be insitituted in the following fashion. First, when $x = o$, $y = b$; hence the following equation will arise : $b = A + B + D$. Secondly, since $\frac{c^4d^2y}{dx^2} = \int dx \int ydx$, and $\int pdx = o$ for $x = o$, hence when $x = o$, $\frac{d^2y}{dx^2} = o$. But

$$\frac{d^2y}{dx^2} = \frac{A}{c^2}e^{\frac{x}{c}} + \frac{B}{c^2}e^{\frac{-x}{c}} - \frac{C}{c^2}\sin\frac{x}{c} - \frac{D}{c^2}\cos\frac{x}{c},$$

whence the second equation appears, namely $A + B - D = o$. Thirdly, since $\frac{c^4d^3y}{dx^3} = \int ydx$, then when $x = o$, $\frac{d^3y}{dx^3}$ disappears,

and $\frac{c^3d^3y}{dx^3} = Ae^{\frac{x}{c}} - Be^{-\frac{x}{c}} - C\cos\frac{x}{c} + D\sin\frac{x}{c}$, whence the third equation $A - B - C = o$ appears. Fourthly, if $x = a$, $y = o$, and $Ae^{\frac{a}{c}} + Be^{-\frac{a}{c}} + C\sin\frac{a}{c} + D\cos\frac{a}{c} = o$. Fifthly, because AB is tangent to the curve at the point B, when $x = a$, $\frac{dy}{dx} = o$, whence the fifth equation $Ae^{\frac{a}{c}} - Be^{-\frac{a}{c}} + C\cos\frac{a}{c} - D\sin\frac{a}{c} = o$ appears. From these five equations the four constants A, B, C, and D will be determined; and that on which the hinge of the matter turns, the value of $c = \sqrt[4]{\frac{Ek^2af}{M}}$ will be found, from which the length of the simple isochronous pendulum will be secured, whereby the durations of the oscillations will become known.

70. From the second and third equation, the constants C and D will be expressed in terms of A and B thusly: $C = A - B$, and $D = A + B$. These values, substituted in the fourth and fifth equations, will give

$$Ae^{\frac{a}{c}} + Be^{-\frac{a}{c}} + (A - B)\sin\frac{a}{c} + (A + B)\cos\frac{a}{c} = o, \text{ and}$$

$$Ae^{\frac{a}{c}} - Be^{-\frac{a}{c}} + (A - B)\cos\frac{a}{c} - (A + B)\sin\frac{a}{c} = o, \text{ from}$$

which we secure

$$\frac{A}{B} = \frac{-e^{-\frac{a}{c}} + \sin\frac{a}{c} - \cos\frac{a}{c}}{e^{\frac{a}{c}} + \sin\frac{a}{c} + \cos\frac{a}{c}} = \frac{e^{-\frac{a}{c}} + \cos\frac{a}{c} + \sin\frac{a}{c}}{e^{\frac{a}{c}} + \cos\frac{a}{c} - \sin\frac{a}{c}},$$

whence the following equation will be obtained:

$$2 + (e^{\frac{a}{c}} + e^{-\frac{a}{c}})\cos\frac{a}{c} = o, \text{ or } e^{\frac{2a}{c}}\cos\frac{a}{c} + 2e^{\frac{a}{c}} + \cos\frac{a}{c} = o.$$

This gives $e^{\frac{a}{c}} = \frac{-1 \pm \sin\frac{a}{c}}{\cos\frac{a}{c}}$.

However, since $e^{\frac{a}{c}}$ is a positive quantity, then $\cos\frac{a}{c}$ will be negative, and the angle $\frac{a}{c}$ will be greater than a right angle.

71. From the last equation it is seen that an infinite number of angles $\frac{a}{c}$ will satisfy it, due to which angles an infinite number of diverse modes of oscillations of the same ribbon arise. For the curve can cut the axis AB at one or more points before it touches the axis at B; from this fact there is an infinite number of modes of oscillations equally possible. Since we are here primarily considering the case in which B is the first point when the ribbon crosses the axis AB, this case will be satisfied by a very small angle $\frac{a}{c}$ satisfying the discovered equation. Since this angle is greater than a right angle, let $\frac{a}{c} = \frac{\pi}{2} + \phi$, in which ϕ is less than a right angle. Hence, because $\sin\frac{a}{c} = \cos\phi$, and $\cos\frac{a}{c} = -\sin\phi$, we shall obtain the double equation $e^{\frac{a}{c}} = \frac{1 \pm \cos\phi}{\sin\phi}$,

which gives either $e^{\frac{a}{c}} = \tan\frac{1}{2}\phi$, or $e^{\frac{a}{c}} = \cot\frac{1}{2}\phi$.
The second equation will give the smaller value for the angle ϕ, and will be adapted to the case proposed.

72. The following possible modes of oscillation will be found if the angle $\frac{a}{c}$ is greater than two right angles and less than three right angles. Let $\frac{a}{c} = \frac{3\pi}{2} - \phi$. Then $\sin\frac{a}{c} = -\cos\phi$, and $\cos\frac{a}{c} = -\sin\phi$, whence

$$e^{\frac{a}{c}} = \frac{1 \pm \cos\phi}{\sin\phi}, \quad \text{or } e^{\frac{a}{c}} = \tan\tfrac{1}{2}\phi, \text{ or } e^{\frac{a}{c}} = \cot\tfrac{1}{2}\phi.$$

In a similar way, other modes of oscillation will be found by letting $\frac{a}{c} = \frac{5\pi}{2} + \phi$; $\frac{a}{c} = \frac{7\pi}{2} - \phi$; etc.

From all these, if the hyperbolic logarithms be taken, there will arise the following equations:

I. $\frac{\pi}{2} + \phi = \log \cot \frac{1}{2}\phi$ II. $\frac{\pi}{2} + \phi = \log \tan \frac{1}{2}\phi$

III. $\frac{3\pi}{2} - \phi = \log \cot \frac{1}{2}\phi$ IV. $\frac{3\pi}{2} - \phi = \log \tan \frac{1}{2}\phi$

V. $\frac{5\pi}{2} + \phi = \log \cot \frac{1}{2}\phi$ VI. $\frac{5\pi}{2} + \phi = \log \tan \frac{1}{2}\phi$

VII. $\frac{7\pi}{2} - \phi = \log \cot \frac{1}{2}\phi$ VIII. $\frac{7\pi}{2} - \phi = \log \tan \frac{1}{2}\phi$

etc.

Now the third of these equations agrees with the second, for let $\frac{1}{2}\phi = \frac{\pi}{2} - \frac{1}{2}\theta$, then $\cot \frac{1}{2}\phi = \tan \frac{1}{2}\theta$, whence the third equation passes over into $\frac{\pi}{2} = \theta = \log \tan \frac{1}{2}\theta$, which is the second equation. In the same way the fourth equation agrees with the first; the fifth and the eighth agree; likewise the sixth agrees with the seventh, etc. Wherefore only the following different equations will appear:

I. $\frac{\pi}{2} + \phi = \log \cot \frac{1}{2}\phi$ II. $\frac{\pi}{2} + \phi = \log \tan \frac{1}{2}\phi$

III. $\frac{5\pi}{2} + \phi = \log \cot \frac{1}{2}\phi$ IV. $\frac{5\pi}{2} + \phi = \log \tan \frac{1}{2}\phi$

V. $\frac{9\pi}{2} + \phi = \log \cot \frac{1}{2}\phi$ VI. $\frac{9\pi}{2} + \phi = \log \tan \frac{1}{2}\phi$

etc.

73. Now the hyperbolic logarithm of any tangent or cotangent of an angle is found by taking the tabular [common] logarithm and subtracting ten from it [23]), and multiplying the remainder by 2.302585092994. In order to shorten the labor, it will be convenient to use logarithms again. Let u be the hyperbolic logarithm of the tangent or cotangent of the angle $\frac{1}{2}\phi$ which is

sought. From the tables let the [common] logarithm of the same tangent or cotangent be taken, and let this value, diminished by ten [23]), be v. Therefore, since $u=2.302585092994\ v$, we shall get, by taking common logarithms, $\log u = \log v + 0.3622156886$. This logarithm being found, since $u = \frac{n\pi}{2} + \phi$, we have $\log u = \log (\frac{n\pi}{2} + \phi)$. To evaluate this, the angle ϕ must be expressed in radians, just as π is expressed in the same fashion, where $\pi = 3.1415926535$, and hence $\frac{\pi}{2} = 1.57079632679$. Now the angle ϕ will be expressed in the same way if it be changed to seconds, and if from the logarithm of this number there be subtracted constantly 5.3144251332 [24]); for thus the logarithm of ϕ will appear, from which, by going back to numbers, the value of ϕ is secured. Now for every kind of oscillation, $\frac{a}{c}$ will constantly equal $u = \frac{n\pi}{2} + \phi$.

74. This advice having been given for instituting the calculations by approximations, the value of the angle ϕ will be secured without difficulty for any kind of oscillation. For by assigning any values you please to ϕ and determining by calculation $\frac{n\pi}{2} + \phi$, and $\log \tan \frac{1}{2}\phi$ or $\log \cot \frac{1}{2}\phi$, soon the approximate value of ϕ will be known.

If now the limits of the angle ϕ be removed as far as you please, then closer limits will be found immediately, and from these the true value of ϕ. Thus for the first equation $\frac{a}{c} = \frac{\pi}{2} + \phi = \log \cot \frac{1}{2}\phi$, I have secured the following limits for the angle ϕ: 17° -26′ and 17° -27′. From these, by the following calculation, the true value of ϕ itself will be obtained.

ϕ	= 17° -26′ -0″	17° -27′ -0″
in sec.	= 62760″	62820″
log	= 4.7976829349	4.7980979321
subtract	= 5.3144251332	5.3144251332

$\log \phi$	$= 9.4832578017 - 10$	$9.4836727989 - 10$
ϕ	$= 0.3042690662$	0.3045599545
$\frac{1}{2}\pi$	$= 1.5707963268$	1.5707963268
$\frac{1}{2}\pi + \phi = \frac{a}{c}$	$= 1.8750653930$	1.8753562813
$\frac{1}{2}\phi$	$= 8^{o}\ -43'\ -0''$	$8^{o}\ -43'\ -30''$
$v = \log \cot \frac{1}{2}\phi$	$= 0.8144034109$	0.8139819342
$\log v$	$= 9.9108395839 - 10$	$9.9106147660 - 10$
add	$= 0.3622156886$	0.3622156886
$\log u$	$= 0.2730552725$	0.2728304546
$u = \frac{a}{c}$	$= 1.8752331540$	1.8742626675
difference	$= +1677610$	-10936138

From these errors of the two limits is is concluded that $\phi = 17^{o}\ -26'\ -7.98''$, and $\frac{1}{2}\pi + \phi = \frac{a}{c} = 107^{o}\ -26'\ -7.98''$.

But since

ϕ	$= 62767.98''$
log	$= 4.7977381525$
subtract	$= 5.3144251332$
$\log \phi$	$= 9.4833130193 - 10$
therefore ϕ	$= 0.3043077545$
add $\frac{\pi}{2}$	$= 1.5707963268$
$\phi + \frac{\pi}{2} = \frac{a}{c}$	$= 1.8751040813$

This being found, then $\frac{A}{B} = \tan \frac{1}{2}\phi = 0.1533390624$ [25]).

Therefore the ratio of the constants A and B is found. From which also the ratio of the remaining constants will be known in relation to them.

75. There still remains the first equation $b = A + B + D$. This equation, since $D = A + B$, becomes $b = 2A + 2B$, and therefore $A + B = \frac{1}{2}b$.

Since therefore $\frac{A}{B} = \tan \frac{1}{2}\phi$, $B\ (1 + \tan \frac{1}{2}\phi) = \frac{1}{2}b$, and

$$B = \frac{b}{2 + 2 \tan \frac{1}{2}\phi} = \frac{b}{2(1 + \tan \frac{1}{2}\phi)}$$

Whence from $\tan \frac{1}{2}\phi = 0.1533390624$ the several constants of the equation will be determined in the following fashion:

$$\frac{A}{b} = \frac{\tan \frac{1}{2}\phi}{2(1 + \tan \frac{1}{2}\phi)} = \frac{0.1533390624}{2.3066781248}$$

$$\frac{B}{b} = \frac{1}{2(1 + \tan \frac{1}{2}\phi)} = \frac{1.0000000000}{2.3066781248}$$

$$\frac{C}{b} = \frac{-1 + \tan \frac{1}{2}\phi}{2(1 + \tan \frac{1}{2}\phi)} = \frac{-0.8466609376}{2.3066781248}$$

$$\frac{D}{b} = \frac{1 + \tan \frac{1}{2}\phi}{2(1 + \tan \frac{1}{2}\phi)} = \frac{1.1533390624}{2.3066781248}$$

These being found, the character of the curve aMB which the ribbon takes during oscillation will be expressed by the following equation:

$$\frac{y}{b} = \frac{A}{b} e^{\frac{x}{c}} + \frac{B}{b} e^{-\frac{x}{c}} + \frac{C}{b} \sin \frac{x}{c} + \frac{D}{b} \cos \frac{x}{c}.$$

76. As to the velocity of the oscillations, it will become known from the equation $\frac{a}{c} = 1.8751040813$. For the sake of brevity, put $n = 1.8751040813$, so that $a = nc$; and since $c^4 = \frac{Ek^2af}{M}$ where $\frac{M}{a}$ expresses the specific gravity of the ribbon and Ek^2 the absolute elasticity, by the method which I have used hitherto, $a^4 = n^4 Ek^2 \frac{af}{M}$, and on that account $f = \frac{a^4}{n^4} \cdot \frac{1}{Ek^2} \cdot \frac{M}{a}$ from which the length of a simple isochronous pendulum will vary directly as the fourth power of the length of the ribbon, directly as the specific gravity, and inversely as the absolute elasticity. Let g be the length of a simple pendulum oscillating in a single second in such a way that $g = 3.16625$ Rhenish feet. Since the durations of the oscillations of the pendulums are proportional to the square roots of the lengths of the pendulums, the time of one oscillation made by our elastic ribbon will be $\frac{\sqrt{f}}{\sqrt{g}}$ seconds $= \frac{a^2}{n^2} \sqrt{\frac{1}{g} \cdot \frac{1}{Ek^2} \cdot \frac{M}{a}}$. Whence the number of oscilla-

tions produced in one second will be

$$\frac{n^2}{a^2}\sqrt{g\,Ek^2\,\frac{a}{M}}$$

This number expresses the pitch of the tone which the ribbon produces. Therefore the sound produced by different elastic ribbons fastened at one end in a wall, will be proportional to the square root of the absolute elasticity, inversely proportional to the square root of the specific gravity, and inversely proportional to the square of the length. Wherefore if two elastic ribbons differ only in length, their tones are inversely proportional to the square of the lengths; in other words, a ribbon twice as long will give forth a tone two octaves lower. Now a tense chord twice as long gives forth a tone only one octave lower, if the tension remains the same. From this it is clear that the tones of elastic ribbons follow a very different ratio from that of the tones in tense chords [26]).

77. As to the character of the curve *aMB* continued beyond the ends *a* and *B*, it is clear, in the first place, that the curve beyond *a* advances in such a way that it is continually diverging from the axis *BA*. For taking *x* as a negative quantity,

$$y = Be^{\frac{x}{c}} + Ae^{-\frac{x}{c}} - \text{Csin}\,\frac{x}{c} + \text{Dcos}\,\frac{x}{c}.$$

Now here, all the limits are positive, because only the coefficient *C* previously had a negative value [sec. 75]; whence while *x* increases, *y* must also increase, because the number *B* is greater than *A*, and so the term $Be^{\frac{x}{c}}$ prevails. Now as soon as $\frac{x}{c}$ has reached even a moderate value, then the term $Be^{\frac{x}{c}}$ increases in such a degree that the remaining terms, in comparison with it, disappear, as it were. For this reason, because the radius of curvature of the curve at *B* does not equal infinity, for $\frac{Ek^2}{R} = \frac{M}{af}\int dx \int y\,dx$ and hence the curve at *B* will not have a point of contraflexure, and will advance further on the same side of the axis *AB*, and by increasing the abscissa *x* beyond $BA = a$, the first term $Ae^{\frac{x}{c}}$ soon becomes so great that the

remaining terms can be regarded as zero in comparison with it.

78. Therefore this is the first mode of oscillation among those innumerable ones to which the same ribbon may adapt itself. The second mode, represented in FIG. 21, whereby the ribbon fixed at B crosses the axis AB at one point O, will be deduced from the equation $\frac{a}{c} = \frac{1}{2}\pi + \phi = \log \tan \frac{1}{2}\phi$ or $\frac{3\pi}{2} - \phi = \log \cot \frac{1}{2}\phi$. Here, by means of certain experiments, I have discovered that the angle is contained between the following limits: $1^0\ 2'\ 40''$ and $1^0\ 3'\ 0''$. From this as above the true value of ϕ itself will be secured.

ϕ	$= 1^0\ 2'\ 40''$	$1^0\ 3'\ 0''$
in sec.	$= 3760''$	$3780''$
log	$= 3.5751878450$	3.5774917998
subtract	$= 5.3144251332$	5.3144251332
$\log \phi$	$= 8.2607627118 - 10$	$8.2630666666 - 10$
ϕ	$= 0.0182289944$	0.0183259571
$\frac{3\pi}{2}$	$= 4.7123889804$	4.7123889804
$\frac{3\pi}{2} - \phi = \frac{a}{c}$	$= 4.6941599860$	4.6940630233
$\frac{1}{2}\phi$	$= 31'\ 20''$	$31'\ 30''$
$\log \cot \frac{1}{2}\phi$	$= 2.0402552577$	2.0379511745
$\log v$	$= 0.3096845055$	0.3091937748
add	$= 0.3622156886$	0.3622156886
$\log u$	$= 0.6719001941$	0.6714094634
u	$= 4.6978613391$	4.6925559924
$\frac{a}{c}$	$= 4.6941599860$	4.6940630233
error	$= +37013531$	-15070309

From these errors the true value of the angle ϕ is found to be $1^0\ 2'\ 54.213''$, and $\frac{a}{c} = 268^0\ 57'\ 5.787''$. Since therefore

ϕ	$= 3774.213''$
log	$= 3.5768264061$
subtract	$= 5.3144251332$

$\log \phi = 8.2624012729 - 10$

$\phi = 0.0182979009$

$3\frac{\pi}{2} = 4.7123889804$

$\frac{3\pi}{2} - \phi = \frac{a}{c} = 4.6940910795$

Therefore the tone of a ribbon oscillating in the first case will be to the tone of the same ribbon vibrating in this case as the square of 1.8751040813 is to the square of 4.6940910795, or as 1 to 6.266891, or, in least integers, as 4 is to 25, or as 1 is to $6\frac{4}{15}$. Whence the latter tone will be just about 2 octaves plus a fifth plus a half tone higher than the former [27]).

79. For the following cases of oscillations of the same ribbon, in which the ribbon cuts the axis AB at two or more points while oscillating, the angle ϕ becomes much smaller; thus for the third case the following equation is secured:

$$\frac{5\pi}{2} + \phi = \log \cot \tfrac{1}{2}\phi = \frac{a}{c}.$$

Therefore, since $e^{\frac{5\pi}{2}+\phi} = \cot \frac{1}{2}\phi$, because ϕ is an extermely small angle,

$$e^{\frac{5\pi}{2}+\phi} = e^{\frac{5\pi}{2}}\left(1 + \phi + \frac{\phi^2}{2} + \frac{\phi^3}{6} + \ldots.\right),$$

$$\text{and } \cot \tfrac{1}{2}\phi = \frac{1 - \frac{1}{8}\phi^2}{\frac{1}{2}\phi - \frac{1}{48}\phi^3} = \frac{2}{\phi} - \frac{\phi}{6}.$$

Hence approximately,

$$e^{\frac{5\pi}{2}} = \frac{2}{\phi}, \text{ or } \phi = 2e^{\frac{-5\pi}{2}},$$

or more closely

$$\phi = \frac{1}{1 + \frac{1}{2}e^{\frac{5\pi}{2}}}$$ [28]), whence $\frac{a}{c} = \frac{5\pi}{2} + \frac{2}{2 + e^{\frac{5\pi}{2}}}$,

The latter term is extremely small. In a similar manner for the fourth case of oscillations, approximately

$$\frac{a}{c} = \frac{7}{2}\pi = 2e^{\frac{-7\pi}{2}}$$

Since these second terms are on the point of vanishing, the values of $\frac{a}{c}$ will be $\frac{9\pi}{2}$, $\frac{11\pi}{2}$, etc., which will differ less from the true values the farther they proceed.

Concerning the Oscillations of a Free Elastic Ribbon

80. Let us now consider an elastic ribbon fixed at no point, but free or lying upon an extremely smooth plane or, neglecting gravity, existing in a vacant space. Now it is readily apparent that a ribbon of this kind can receive an oscillatory motion, while the ribbon *acb* (Fig. 22), curving itself, passes alternately on one side and the other side of the position of rest AB. Therefore the oscillatory motion may be defined in the same way in which it was defined in the preceding case, provided only that the calculations be adapted to this case in the necessary manner. Therefore let *acb* be the curved shape of the ribbon which it assumes while oscillating, and ACB the shape of the same ribbon in the state of equilibrium through which it passes in each oscillation. As before, let the length of the ribbon $AB = a$, the absolute elasticity be Ek^2, and the weight or mass equal to M. Then let the abscissa $AP = x$, the ordinate $PM = y$, the arc $aM = s$, which will correspond with the abscissa x in such a way that $ds = dx$; from this the radius of curvature at M will be $R = \frac{dx^2}{d^2y}$. Further, let the first ordinate $Aa = b$. All this being stipulated, by instituting the same process of reasoning as before [sec. 66 and 67], we shall arrive at the same equation

$$\frac{E\,k^2}{R} = \frac{M}{af}\int dx \int y\,dx = \frac{E\,k^2\,d^2\,y}{dx^2}$$

81. Therefore, if we take $\frac{E\,k^2\,af}{M} = c^4$, where f, as before, expresses the length of a simple isochronous pendulum, we shall have, by integrating, the following equation for the curve:

$$y = Ae^{\frac{x}{c}} + Be^{\frac{-x}{c}} + C\sin\frac{x}{c} + D\cos\frac{x}{c}.$$

This will be adapted to the present case as follows:
First, when $x = o, y = b$, and hence $b = A + B + D$. Second.

since $\frac{c^4\,d^2y}{dx^2} = \int dx \int y dx$, then $\frac{d^2y}{dx^2} = o$ when $x = o$, whence arises $A + B - D = o$. Thirdly, since $\frac{c^4\,d^3y}{dx^3} = \int y dx$, then $\frac{d^3y}{dx^3} = o$ when $x = o$, whence $A - B - C = o$. Fourthly, if $x = a$, $\int y dx$, or $\frac{d^3y}{dx^3}$ must vanish, because $\int y dx$ expresses the sum of all the forces drawing the ribbon in a direction normal to the axis AB, and if this sum were not equal to zero, the ribbon itself would undergo a local motion contrary to the conditions instituted; for this reason, therefore,

$$Ae^{\frac{a}{c}} - Be^{\frac{-a}{c}} - \text{Ccos}\,\frac{a}{c} + \text{Dsin}\,\frac{a}{c} = \text{o}.$$

Fifth, since the ribbon is free at the end B, it cannot have any curvature there, and therefore $\frac{d^2y}{dx^2} = o$ when $x = a$, whence

$$Ae^{\frac{a}{c}} + Be^{\frac{-a}{c}} - \text{Csin}\,\frac{a}{c} - \text{Dcos}\,\frac{a}{c} = \text{o}.$$

By taking these five conditions into the computations, not only will the four constants A, B, C, and D be determined, but also the value of the fraction $\frac{a}{c}$ will be found; from which the length of the simple isochronous pendulum f will become known.

82. From the second and third of these equations, $D = A + B$, and $C = A - B$, and these values, substituted in the equations above, will give the following:

$$Ae^{\frac{a}{c}} - Be^{\frac{-a}{c}} - \text{(A—B) cos}\,\frac{a}{c} + \text{(A+B) sin}\,\frac{a}{c} = \text{o, and}$$

$$Ae^{\frac{a}{c}} + Be^{\frac{-a}{c}} - \text{(A—B) sin}\,\frac{a}{c} - \text{(A+B) cos}\,\frac{a}{c} = \text{o}.$$

From these it is found that

$$\frac{A}{B} = \frac{e^{\frac{-a}{c}} - \cos\frac{a}{c} - \sin\frac{a}{c}}{e^{\frac{a}{c}} - \cos\frac{a}{c} + \sin\frac{a}{c}} = \frac{-e^{\frac{-a}{c}} - \sin\frac{a}{c} + \cos\frac{a}{c}}{e^{\frac{a}{c}} - \sin\frac{a}{c} - \cos\frac{a}{c}}.$$

from which the following equation is secured:

$$2 - e^{\frac{a}{c}} \cos\frac{a}{c} - e^{\frac{-a}{c}} \cos\frac{a}{c} = 0, \text{ or } e^{\frac{a}{c}} = \frac{1 \pm \sin\frac{a}{c}}{\cos\frac{a}{c}}.$$

Whence the following equations will be found:

I. $$\frac{a}{c} = \frac{\pi}{2} - \phi = \log \tan \tfrac{1}{2}\phi$$

which gives $\frac{a}{c} = o$ for the natural position of the ribbon [29]),

II. $$\frac{a}{c} = \frac{\pi}{2} - \phi = \log \cot \tfrac{1}{2}\phi$$

III. $$\frac{a}{c} = \frac{3\pi}{2} + \phi = \log \cot \tfrac{1}{2}\phi$$

IV. $$\frac{a}{c} = \frac{5\pi}{2} - \phi = \log \cot \tfrac{1}{2}\phi$$

V. $$\frac{a}{c} = \frac{7\pi}{2} + \phi = \log \cot \tfrac{1}{2}\phi$$

VI. $$\frac{a}{c} = \frac{9\pi}{2} - \phi = \log \cot \tfrac{1}{2}\phi$$

etc.

83. These equations again indicate innumerable modes of oscillations. In the second of these the ribbon will cut the axis AB only once; in the third, twice; in the fourth, three times, in the fifth, four times, and so on. From this it is understood that the second, fourth, sixth, etc. modes are not adaptable to the present case. For since in these the number of intersections is uneven, the position of the ribbon, while oscillating in the second mode, would be such as FIG. 23 represents, in which mode, however small the sum of the forces acting throughout the whole ribbon tends to become, nevertheless the ribbon would acquire from them a rotary motion around the center point C, because the forces applied to each half aC and bC would combine to produce the same rotary motion in the ribbon. For this reason, since the rotary motion must be absolutely excluded, the shape of the ribbon which is taken on during oscillation ought to be of such a character that not only the sum of the acting forces

applied to the whole ribbon equals zero, but also that the sum of their moments tends to vanish; and this is obtained if the curve at the center point c (FIG. 22) be given a diameter cC. This takes place if the curve cuts the axis AB in two, in four, or in general, an even number of points; from which the 3rd, 5th, 7th, etc., equations only, will give us satisfactory solutions [30]).

84. This limitation will be found to be contained in the very statement of the problem if we admit only curves of the kind that have the straight line Cc as a diameter, that is, in which the value of y would be the same if $(a - x)$ should be written in place of x. Therefore let us substitue $(a - x)$ in place of x in the general equation, whence

$$y = Ae^{\frac{a}{c}}\, e^{\frac{-x}{c}} + Be^{\frac{-a}{c}}\, e^{\frac{x}{c}} + C\sin\frac{a}{c}\cos\frac{x}{c} - C\cos\frac{a}{c}\sin\frac{x}{c} + D\cos\frac{a}{c}\cos\frac{x}{c} + D\sin\frac{a}{c}\sin\frac{x}{c}.$$

Since this equation must agree with the equation

$$y = Ae^{\frac{x}{c}} + Be^{\frac{-x}{c}} + C\sin\frac{x}{c} + D\cos\frac{x}{c},$$

therefore

$$Ae^{\frac{a}{c}} = B,\ C\left(1 + \cos\frac{a}{c}\right) = D\sin\frac{a}{c},\ \text{and}\ C\sin\frac{a}{c} = D\left(1 - \cos\frac{a}{c}\right).$$

The last two equations are identical. Since therefore $\frac{A}{B} = e^{\frac{-a}{c}}$ when this value is compared with the expression in section 82, there will appear

$$e^{\frac{-a}{c}} - \cos\frac{a}{c} - \sin\frac{a}{c} = 1 - e^{\frac{-a}{c}}\cos\frac{a}{c} + e^{\frac{-a}{c}}\sin\frac{a}{c},\ \text{or}$$

$$e^{\frac{-a}{c}} = \frac{1 + \cos\frac{a}{c} + \sin\frac{a}{c}}{1 + \cos\frac{a}{c} - \sin\frac{a}{c}} = \frac{1 + \sin\frac{a}{c}}{\cos\frac{a}{c}} = \frac{\cos\frac{a}{c}}{1 - \sin\frac{a}{c}}.$$

85. Therefore $e^{\frac{a}{c}} = \dfrac{1 - \sin\frac{a}{c}}{\cos\frac{a}{c}}$.

All this is contained in the equation previously found [sec. 82].

$$e^{\frac{a}{c}} = \frac{1 + \sin\frac{a}{c}}{\cos\frac{a}{c}}$$

Merely one half of the cases shown above [end of sec. 82], in other words those which have uneven numbers, will state the present problem. Wherefore, since the first equation comprises the natural state of the ribbon, all the modes of oscillations will be comprised in the following equations:

I. $\dfrac{a}{c} = \dfrac{3\pi}{2} + \phi = \log\cot\frac{1}{2}\phi$

II. $\dfrac{a}{c} = \dfrac{7\pi}{2} + \phi = \log\cot\frac{1}{2}\phi$

III. $\dfrac{a}{c} = \dfrac{11\pi}{2} + \phi = \log\cot\frac{1}{2}\phi$

etc.

Therefore the first of these equations will give us the first and principal mode of oscillation, for which the value of the angle will be found by approximation, in a way similar to that used before. Now the limits of the angle ϕ are soon found to be $1^0\text{-}0'\text{-}40''$ and $1^0\text{-}1'\text{-}0''$, from which, by the following calculation, the true value of ϕ is secured.

ϕ	$= 1^0\text{-}0'\text{-}40''$	$1^0\text{-}1'\,0''$
in sec.	$= 3640''$	$3660''$
log	$= 3.5611013836$	3.5634810854
subtract	$= 5.3144251332$	5.3144251332
$\log\phi$	$= 8.2466762504 - 10$	$8.2490559522 - 10$
ϕ	$= 0.0176472180$	0.0177441807
$3\frac{\pi}{2}$	$= 4.7123889804$	4.7123889804
$3\frac{\pi}{2} + \phi = \frac{a}{c}$	$= 4.7300361984$	4.7301331611
$\frac{1}{2}\phi$	$= 30'\text{-}20''$	$30'\text{-}30''$
v	$= 2.0543424742$	2.0519626482

$\log v$	$= 0.3126728453$	0.3121694510
add	$= 0.3622156886$	0.3622156886
$\log u$	$= 0.6748885339$	0.6743851396
$u = \frac{a}{c}$	$= 4.7302983543$	4.7248186037
error	$= + 636341$	$+ 53145574$
		636341
	Difference	52509233

From this it is seen that the true value of ϕ is not contained between those limits, but is somewhat less than $1^0\text{-}0'\text{-}40''$ [31]). None the less, however, it will be found from these errors. For let $\phi = 1^0\text{-}0'\text{-}40'' - n''$; then

$$20'' : 52509233 :: n'' : 636341, \text{ whence}$$

$$n = \frac{2423}{10{,}000}, \text{ and}$$

ϕ	$= 1^0\text{-}0'\text{-}39.7576''$
or	$= 3639.7576''$
log	$= 3.5610724615$
subtract	$= 5.3144251332$
$\log \phi$	$= 8.2466473283 - 10$
ϕ	$= 0.0176460428$
$3\frac{\pi}{2}$	$= 4.7123889804$
$3\frac{\pi}{2} + \phi = \frac{a}{c}$	$= 4.7300350232$ [correct value 4.7300408]

86. Let this number be equal to m, so that

$$c^4 = \frac{Ek^2af}{M}, \text{ and so that } a^4 = \frac{m^4Ek^2af}{M}, \text{ and } f = \frac{a^4}{m^4}\,\frac{1}{Ek^2}\cdot\frac{M}{a}.$$

Hence in the same way the number of oscillations produced by this ribbon in a single second will be

$$\frac{m^2}{a^2}\sqrt{g\,Ek^2\frac{a}{M}}$$

where $g = 3.16625$ Rhenish feet. Now if the same ribbon be made to produce a tone when it is either free, or has one end B fixed in a wall, the tones will be in the ratio of $n^2 : m^2$, or as the

square of the numbers 1.8751040813 and 4.7300350232 [correct value 4.7300408], or as 1 is to 6.363236. The ratio of these tones will be approximately 11 : 70. Therefore the interval between these tones will be two octaves plus a fifth plus a half tone. If the free ribbon be taken twice the length of the fixed ribbon, the interval between the tones will be about a minor sixth. $\left[\frac{8}{5} = \frac{72}{45} \text{ instead of } \frac{70}{44}\right]$

87. The value for the fraction $\frac{a}{c}$ being found, the equation for the curve which the ribbon forms during oscillations, hitherto indeterminate, may now be determined; for

$$e^{\frac{a}{c}} = \frac{1 - \sin\frac{a}{c}}{\cos\frac{a}{c}}, \text{ and } Ae^{\frac{a}{c}} = B; \text{ hence } B = \frac{1 - \sin\frac{a}{c}}{\cos\frac{a}{c}}A,$$

$$C = A - B = \frac{A\left(\cos\frac{a}{c} + \sin\frac{a}{c} - 1\right)}{\cos\frac{a}{c}},$$

$$D = A + B = \frac{A\left(\cos\frac{a}{c} - \sin\frac{a}{c} + 1\right)}{\cos\frac{a}{c}},$$

Now

$$b = A + B + D = 2D = \frac{2A\left(\cos\frac{a}{c} - \sin\frac{a}{c} + 1\right)}{\cos\frac{a}{c}},$$

whence

$$A = \frac{b\cos\frac{a}{c}}{2\left(\cos\frac{a}{c} - \sin\frac{a}{c} + 1\right)} = \frac{b\left(+1 + \sin\frac{a}{c} - \cos\frac{a}{c}\right)}{4\sin\frac{a}{c}}$$

$$B = \frac{b\left(1 - \sin\frac{a}{c}\right)}{2\left(\cos\frac{a}{c} - \sin\frac{a}{c} + 1\right)} = \frac{b\left(-1 + \sin\frac{a}{c} + \cos\frac{a}{c}\right)}{4\sin\frac{a}{c}}$$

$$C = \frac{b\left(-1 + \sin\frac{a}{c} + \cos\frac{a}{c}\right)}{2\left(\cos\frac{a}{c} - \sin\frac{a}{c} + 1\right)} = \frac{b\left(1 - \cos\frac{a}{c}\right)}{2\sin\frac{a}{c}}$$

$$D = \frac{b}{2} = \frac{b\sin\frac{a}{c}}{2\sin\frac{a}{c}}.$$

These being substituted, the following equation will arise:

$$\frac{y}{b} = \frac{e^{\frac{x}{c}}\cos\frac{a}{c} + e^{-\frac{x}{c}}\left(1 - \sin\frac{a}{c}\right)}{2\left(1 - \sin\frac{a}{c} + \cos\frac{a}{c}\right)} +$$

$$\frac{(1-\cos\frac{a}{c})\sin\frac{x}{c}+\sin\frac{a}{c}\cos\frac{x}{c}}{2\sin\frac{a}{c}}$$

88. Now because the straight line Cc is the diameter of the curve, let there be taken, from the central point C, the abscissa $CP = z$, then $x = \frac{1}{2}a - z$, whence

$$e^{\frac{x}{c}} = e^{\frac{a}{2c}}\, e^{\frac{-z}{c}}\sqrt{\frac{1-\sin\frac{a}{c}}{\cos\frac{a}{c}}}, \text{ and } e^{\frac{-x}{c}} = e^{\frac{z}{c}}\sqrt{\frac{\cos\frac{a}{c}}{1-\sin\frac{a}{c}}},$$

whence

$$\frac{Ae^{\frac{x}{c}}+Be^{\frac{-x}{c}}}{b} = \frac{(e^{\frac{z}{c}}+e^{\frac{-z}{c}})\sqrt{\cos\frac{a}{c}\,(1-\sin\frac{a}{c})}}{2\,(1-\sin\frac{a}{c}+\cos\frac{a}{c})} = \frac{e^{\frac{z}{c}}+e^{\frac{-z}{c}}}{2\,(e^{\frac{a}{2c}}+e^{\frac{-a}{2c}})}$$

$$\textit{Further}\,(1-\cos\frac{a}{c})\sin\frac{x}{c}+\sin\frac{a}{c}\cos\frac{x}{c} = \sin\frac{x}{c}+\sin\frac{a-x}{c}$$

$$= \sin(\frac{a}{2c}-\frac{z}{c})+\sin(\frac{a}{2c}+\frac{z}{c}).$$

$$= 2\sin\frac{a}{2c}\cos\frac{z}{c}.$$

These values being substituted, the following equation will arise :

$$\frac{2y}{b} = \frac{e^{\frac{z}{c}}+e^{\frac{-z}{c}}}{e^{\frac{a}{2c}}+e^{\frac{-a}{2c}}} + \frac{\cos\frac{z}{c}}{\cos\frac{a}{2c}}$$

which is the most simple form whereby the character of the curve $aMcb$ can be expressed. Now it is manifest that whether z be taken as positive or negative, the same value for y will appear.

It is also true that $e^{\frac{a}{2c}}+e^{\frac{-a}{2c}} = -\frac{2\cos\frac{a}{2c}}{\sqrt{\cos\frac{a}{c}}}$ [32])

We have found that the angle $\frac{a}{c} = 271^{\circ}\ 0'\ 39\ 3/4''$ [correct value is $271^{\circ}\ 0'\ 40.94''$].

89. Setting $z = o$, y will express the value of the ordinate Cc. This gives

$$\frac{2Cc}{b} = -\frac{2\sqrt{\cos\frac{a}{c}}}{2\cos\frac{a}{2c}} + \frac{1}{\cos\frac{a}{2c}} \text{ or } \frac{Cc}{Aa} = \frac{1-\sqrt{\cos\frac{a}{c}}}{2\cos\frac{a}{2c}}$$

$$= \frac{1}{2}\sec\frac{a}{2c} - \frac{1}{2}\sec\frac{a}{2c}\sqrt{\cos\frac{a}{c}}.$$

But $\cos\frac{a}{c} = \sin 1^{0}\text{-}0'\text{-}39\ 3/4''$ [correct value $1^{0}\text{-}0'\text{-}40.94''$]

and $\cos\frac{a}{2c} = -\sin 45^{0}\text{-}30'\text{-}19\ 7/8''$ [correct value $45^{0}\text{-}30'\text{-}20.47''$].

Hence it is found that $\frac{Cc}{Aa} = -0.607815$ [correct value -0.607841].

Then if $y = o$, the points E and F at which the curve intersects the axis will be found; therefore

$$e^{\frac{z}{c}} + e^{-\frac{z}{c}} = -\frac{\cos\frac{z}{c}}{\cos\frac{a}{2c}}\left(e^{\frac{a}{2c}} + e^{-\frac{a}{2c}}\right) = \frac{2\cos\frac{z}{c}}{\sqrt{\cos\frac{a}{c}}},$$

from which, by approximation

$$\frac{CE}{CA} = 0.551685, \text{ and } \frac{AE}{AC} = 0.448315.$$

Therefore, while the ribbon is performing these oscillations, these points E and F will remain motionless. Therefore the oscillatory motion of this kind, which otherwise, it would seem, could scarcely be produced in reality, can be easily produced. For if the ribbon should be fixed at the points E and F defined in this fashion, it would oscillate exactly as if it were free.

90. If the second of the equations found above, viz. $\frac{a}{c} = \frac{7\pi}{2} + \phi = \log\cot\frac{1}{2}\phi$ be treated in this same fashion, in which case ϕ will be approximately zero, then the second mode by which a free ribbon can perform vibrations will appear, that is, by cutting the axis AB at four points. Then the ribbon will oscillate precisely as if it had been fixed at these four points. Conversely, therefore, if the ribbon be fixed at these four points, or merely at any two of them, it will oscillate just as if it were free, and it will produce a much higher tone, inasmuch as it will be in about the same ratio to the preceding tone as $7^{2} : 3^{2}$; that is, the interval will be of two octaves plus a fourth plus

the half of a semitone. The third mode of oscillation, in which $\frac{a}{c} = \frac{11\pi}{2} + \phi = \log \cot \frac{1}{2}\phi$, will have six intersections of the curve *acb* with the axis *AB*. The tone produced will be higher by one octave plus a minor third, $[\frac{121}{49} = 2\frac{2}{5}$, approximately] and the ribbon will produce this tone if it be fixed at two of the six points. Hence it is clear how different tones can be produced by the same ribbon, according to the different ways in which it is fixed at two points; and if the two points at which it is fixed coincide with its intersections in the first, second, or third, etc. modes, the oscillations adapt themselves to some one of the following modes down to an infinite value. In the latter case the tone will be so high that it cannot be heard at all, or what amounts to the same thing, the ribbon will be absolutely unable to take on an oscillatory motion; or at all events, as in the case of a vibrating chord under which a bridge is so placed that its parts have no rational ratio to one another, an indistinct tone will be produced.

On Oscillations of an Elastic Ribbon Fixed at Both Ends

91. Now let the elastic ribbon be fixed at both ends *A* and *B* (Fig. 24), but in such a way that the tangents of the curve at these points are not fixed. To produce this case in experiment, let extremely sharp points *Aα*, and B*β* be fixed to the extremities of the ribbon; these sharp points, when fastened to a wall, will render the extremities *A* and *B* of the ribbon immovable. In order to investigate the oscillatory motion of this elastic ribbon, let us take, as above, the absolute elasticity of the ribbon to be equal to Ek^2, its length $AB = a$, its weight equal to M, and the length of the simple isochronous pendulum equal to f. Let *AMB* be a curvilinear figure which the ribbon takes on while performing oscillations, and let the abscissa $AP = AM = x$, the ordinate $PM = y$, and the radius of curvature at *M* be equal to R. Furthermore, let P be the force which the sharp point *Aα* supports in the direction *Aα*. Because the force by which the element *Mm* must be acted upon in the direction *Mμ* in order

that the ribbon be kept in this position, is equal to $\frac{Mydx}{af}$, the following equation, by the rules described above [sec. 57, 66, 67], will result : $\frac{Ek^2}{R} = Px - \frac{M}{af} \int dx \int ydx.$

But $R = \frac{-dx^2}{d^2y}$, because the curve is concave to the axis; hence

$$\frac{Ek^2d^2y}{dx^2} = \frac{M}{af} \int dx \int ydx - Px.$$

Therefore, when $x = o$, the radius of curvature R at A will be infinite, that is, $d^2y = o$.

92. If this equation be differentiated twice, the same equation which we have found in the preceding case will appear, namely

$$Ek^2d^4y = \frac{M}{af} ydx^4.$$

But if $\frac{Ek^2af}{M}$ be put equal to c^4, the integral equation will be

$$y = Ae^{\frac{x}{c}} + Be^{\frac{-x}{c}} + \text{Csin}\frac{x}{c} + \text{Dcos}\frac{x}{c}.$$

To determine this, let $x = o$, and since $y = o$ at the same time, then $A + B + D = o$. Second, let $x = a$, and since y again must be zero,

$$Ae^{\frac{a}{c}} + Be^{\frac{-a}{c}} + \text{Csin}\frac{a}{c} + \text{Dcos}\frac{a}{c} = o.$$

Third, since $\frac{d^2y}{dx^2}$ must vanish when $x = o$, and when $x = a$, we have $A + B - D = o$, and

$$Ae^{\frac{a}{c}} + Be^{\frac{-a}{c}} - \text{Csin}\frac{a}{c} - \text{Dcos}\frac{a}{c} = o.$$

Now the equations $A + B - D = o$, and $A + B + D = o$, give $D = o$, and $B = -A$, which values, when substituted in the remaining two equations, give

$$A(e^{\frac{a}{c}} - e^{\frac{-a}{c}}) + \text{Csin}\frac{a}{c} = o, \text{ and } A(e^{\frac{a}{c}} - e^{-\frac{a}{c}}) - \text{Csin}\frac{a}{c} = o.$$

These equations cannot be satisfied unless $A = o$, since it is not

possible for $e^{\frac{a}{c}} = e^{\frac{-a}{c}}$, except in the case when $\frac{a}{c} = o$. Then indeed $C\sin\frac{a}{c} = o$, and here, since C cannot be put equal to zero, because there would be no oscillatory motion, $\sin\frac{a}{c}$ will be equal to zero. Therefore $\frac{a}{c} = \pi$, or $\frac{a}{c} = 2\pi$, etc., whence again there arises an infinite number of modes of oscillation, according as the curve AMB cuts the axis either nowhere except at the end points A and B, or at one point, or at two points, or at several points. This is deduced from the equation $y = C\sin\frac{x}{c}$; and however many points of intersection there are, they will be at equal intervals from one another.

93. Since, therefore, for the first and principal mode of oscillation $\frac{a}{c} = \pi$, $a^4 = c^4\pi^4 = \pi^4 Ek^2\frac{af}{M}$, whence $f = \frac{a^4}{\pi^4}\frac{1}{Ek^2}\frac{M}{a}$. Wherefore as far the length of the ribbon is concerned, the tones again will be inversely proportional to the square of the length [sec. 76]. Now the tones of this ribbon, produced in this fashion, will be to the tones of the same ribbon, if it be fastened at one end B in a wall, as π^2 is to the square of the number 1.875104081 3, that is, as 2.807041 is to 1, or, in least integers, as 160 is to 57, an interval which is about one octave plus the third half tone. If the oscillations are related according to the second mode, in which $\frac{a}{c} = 2\pi$, the tone will be higher by 2 octaves, but if $\frac{a}{c} = 3\pi$, the tones will be higher by 3 octaves and a whole tone than in the case where $\frac{a}{c} = \pi$, and so on [33]). In order to adapt this more readily to experiment, it must be noted that here extremely small oscillations must be taken, so that there is no essential elongation of the ribbon. Wherefore, since the tenacity of the ribbon, by which it resists even a slight extension, without which oscillations of this kind cannot be

produced, introduces an alteration here, those points ought to be fixed in such a way that such a minute extension is not impeded. This results if they rest on a perfectly smooth plane. Thus the elastic ribbon AB, equipped at A and B with the cusps $A\alpha$ and $B\beta$, if these cusps be placed upon a mirror, will give a sound which conforms to the calculations.

On Oscillations of an Elastic Ribbon Fastened at Both Ends in a Wall

94. The preceding case having been cleared up, let the discussion of elastic ribbons come to a close with the oscillatory motion of an elastic ribbon fastened in a wall at both ends A and B (Fig. 25), in such a way that during oscillations, not only do the points A and B remain motionless, but also the straight line AB is constantly tangent to the curve AMB at the points A and B.

Here we must again be careful that the bolts fastening A and B are not absolutely firm, but allow as much extension as is required for curvature. Whatsoever be the forces requisite to hold the band fixed at the points A and B, therefore, we shall arrive at the following differential equation of the 4th order :

$$Ek^2d^4y = \frac{M}{af}\, ydx^4.$$

the integral of which is, as above, letting $\frac{Ek^2af}{M} = c^4$.

$$y = Ae^{\frac{x}{c}} + Be^{\frac{-x}{c}} + \mathrm{Csin}\,\frac{x}{c} + \mathrm{Dcos}\,\frac{x}{c}.$$

95. The constants A, B, C, and D must be so defined that, taking $x = o$, not only y disappears, but also dy becomes zero, because at A the curve is tangent to the curve AB. Now the same thing must also take place if $x = a$, whence the following four equations will arise :

I. $A + B + D = o.$

II. $A - B + C = o.$

III. $Ae^{\frac{a}{c}} + Be^{\frac{-a}{c}} + \mathrm{Csin}\,\frac{a}{c} + \mathrm{Dcos}\,\frac{a}{c} = 0$

IV. $Ae^{\frac{a}{c}} - Be^{\frac{-a}{c}} + C\cos\frac{a}{c} - D\sin\frac{a}{c} = 0.$

From the first and second of these equations, it follows that $C = -A + B$, and $D = -A - B$, which values, substituted in the other two equations, will give

$$Ae^{\frac{a}{c}} + Be^{\frac{-a}{c}} - (A - B)\sin\frac{a}{c} - (A + B)\cos\frac{a}{c} = 0, \text{ and}$$

$$Ae^{\frac{a}{c}} + Be^{\frac{-a}{c}} - (A - B)\cos\frac{a}{c} + (A + B)\sin\frac{a}{c} = 0,$$

The sum and difference of these give, respectively,

$$Ae^{\frac{a}{c}} + B\sin\frac{a}{c} - A\cos\frac{a}{c} = 0, \text{ or } \frac{A}{B} = \frac{\sin\frac{a}{c}}{\cos\frac{a}{c} - e^{\frac{a}{c}}},$$

and $$Be^{\frac{a}{c}} - A\sin\frac{a}{c} - B\cos\frac{a}{c} = 0, \text{ or } \frac{A}{B} = \frac{e^{\frac{-a}{c}} - \cos\frac{a}{c}}{\sin\frac{a}{c}},$$

whence

$$2 = (e^{\frac{a}{c}} + e^{\frac{-a}{c}})\cos\frac{a}{c}, \text{ or } e^{\frac{a}{c}} = \frac{1 \pm \sin\frac{a}{c}}{\cos\frac{a}{c}}.$$

This equation, since it agrees with the one found in section 82, will be satisfied by the following solutions, infinite in number:

I. $\frac{a}{c} = \frac{\pi}{2} - \phi = \log\cot\frac{1}{2}\phi$ II. $\frac{a}{c} = \frac{3\pi}{2} + \phi = \log\cot\frac{1}{2}\phi$

III. $\frac{a}{c} = \frac{5\pi}{2} - \phi = \log\cot\frac{1}{2}\phi$ IV. $\frac{a}{c} = \frac{7\pi}{2} + \phi = \log\cot\frac{1}{2}\phi$

etc.

96. It is impossible to satisfy the first of these equations unless $\frac{a}{c} = 90^0$, and therefore $\frac{a}{c} = 0$ [29]; whence the first mode of oscillation arises from the equation $\frac{a}{c} = \frac{3\pi}{2} + \phi = \log\cot\frac{1}{2}\phi$, and since this has been treated above [sec. 85], $\frac{a}{c} = 4.7300350232$ [correct value 4.7300408]. Wherefore the elastic ribbon, both ends of which are held fast in a wall, will make its vibrations exactly as if it were absolutely free. Now this agreement concerns only the first mode of oscillation [34]; for the second mode of

oscillation, in which $\frac{a}{c} = \frac{5\pi}{2} - \phi = \log \cot \frac{1}{2}\phi$, and the ribbon cuts the axis AB at one point during the oscillation, does not have its equivalent in a free ribbon. The third mode of a ribbon fastened at both ends will agree with the second mode of a free ribbon, and so on.

97. The latter two kinds of oscillations [sec. 91 and 94] cannot, for the reason given, be investigated in any suitable fashion by experiment. The first kind, however, [sec. 65] is not only extremely well suited for experiment, but also it can be applied to the study of the absolute elasticity, which we have called Ek^2, of any proposed ribbon. If the tone which a ribbon of this kind produces when fastened at one end in a wall be noted, and a similar tone be produced at the same time in a chord, the number of oscillations produced in a second will become known. If this number be put equal to the expression $\frac{n^2}{a^2}\sqrt{g\frac{Ek^2a}{M}}$, since n is known, and the quantities g, a, and M have been found by measurements, then the value of the expression Ek^2 will become known, and so also the absolute elasticity. This latter value can be compared with that absolute elasticity which we have already shown how to find from the curvature. [35]) [sec. 38]

NOTES ON THE MONOGRAPH OF

LEONHARD EULER

Concerning Elastic Curves

1744

Additamentum I to the ad Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes.

(Original notes by H. LINSENBARTH in " Ostwald's Klassiker der exakten Wissenschaften " number 175. Translation and changes from the original German by DONALD M. BROWN.)

For a pertinent understanding of the older, fundamental works on elastics, it is necessary to know the connections of the statements contained in them with the methods of the Mechanics of Solids and the Mechanics of Continua.

The following introductory remarks should make possible, even to those readers who have not thoroughly studied general mechanics, a critical judgment of the most important of the original works on this subject.

a) Suppose the originally straight elastic wire (lamina) to be replaced by a chain of infinitely small, stiff body elements, which are connected with one another by spigot-joints, all of whose axes are and remain perpendicular to a fixed plane. At the joints C and C' of any element K, the forces $-\bar{r}$ and $\bar{r}'$ are transmitted from the preceding and following elements respectively. In addition to these single forces, the moments $-\bar{R}$ and $\bar{R}'$ will be transmitted at C and C', if the joints offer resistance to rotation of the elements about the spigot-joint axes. Let the resultant of the external forces acting on K be $d\bar{k}$, and its moment with respect to the point of rotation C of the element K be $d\bar{M}$. The moments of reaction $\bar{R}$ and $\bar{R}'$ can be referred to a suitable pole O in the plane.

Putting $\overline{OC} = \bar{c}$, $\overline{OC'} = \bar{c}'$, $\bar{c}' - \bar{c} = d\bar{c}$, $\bar{r}' - \bar{r} = d\bar{r}$, and $\bar{R}' - \bar{R} = d\bar{R}$, the principles of elementary statics give, for the equilibrium of the forces on the body element K, the conditions :

$$d\bar{r} + d\bar{k} = o, \quad \text{and} \quad d\bar{R} + d\bar{c}.\bar{r} + d\bar{M} = o \text{ (Pole at } C).$$

Introducing the element of arc ds of the equilibrium curve (axis of the curved inextensible wire), which connects the joint-point C, the specific quantities $\bar{\kappa} = \dfrac{d\bar{k}}{ds}$ and $\bar{m} = \dfrac{d\bar{M}}{ds}$, which are related to the constant length of the axis, may be used instead of the absolute quantities $d\bar{k}$ and $d\bar{M}$. In addition, taking $\dfrac{d\bar{c}}{ds} = \bar{\sigma}$, then the static relation takes the form 1) $\dfrac{d\bar{r}}{ds} + \bar{\kappa} = o$, and 2) $\dfrac{d\bar{R}}{ds} + \bar{\sigma}\bar{r} + \bar{m} = o$.

These equations appear frequently in modern literature, since they are not present in this explicit form in the works of the older writers (JACOB BERNOULLI, EULER). EULER knew these equations in their corresponding form for chains whose links are of finite dimensions (compare the statements by ROUTH in his "Dynamik," German edition, vol. 2, page 71.) He merely failed to take the transitional step. Compare also EULER's statement in section 57 (page 116).

Equations 1) and 2) are advanced by CLEBSCH, "Elastizität fester Körper," Leipzig 1862, pp. 204-222, and were used to establish KIRCHHOFF's theory of wires. They are also found in THOMSON and TAIT "Natural Philosophy" Part 2, 1st ed., Oxford, 1867; 2d ed., Cambridge, 1895, pp. 152-155; in LOVE "Theory of Elasticity," 2d ed. Cambridge, 1906, pp. 370-372; and, in their direct relation to the theory of body chains with finite links, in the "Zeitschrift für Math. und Phys." vol. 56, 1908, pp. 68 ff. by K. HEUN. From a general point of view, they have been treated in great detail by E. and F. COSSERAT in "Théorie des corps déformables" Paris 1909, pp. 6-65.

b) Let the departure of the axis element $d\bar{c}$ in the direction of the x-axis of a fixed set of axes Oxy be defined by the angle θ, and let the contingent angle of the elastic curve be denoted in magnitude and direction by $d\bar{\theta}$. By this stipulation, $\dfrac{d\bar{\theta}}{ds}$ is the specific rotation of the axis element $d\bar{c}$. In addition to the direction of the tangent ($\bar{\sigma}$), let the direction of the normal to the curve, $\bar{\nu}$,

be introduced. Then $\bar{\eta} = \overline{\sigma\nu}$ is the binormal (of unit length) on the plane curve, so that $\frac{d\bar{\theta}}{ds} = \bar{\omega} = \frac{1}{a_1}\bar{\eta}$, where a_1, is the radius of curvature of the elastic curve.

According to the hypothesis of DANIEL BERNOULLI, $\bar{R} = P\bar{\omega}$, where P is a constant depending upon the dimensions of the cross section and the coefficient of elasticity.

Further, let $$\kappa_x = \frac{\partial u}{\partial \bar{c}_x}, \kappa_y = \frac{\partial u}{\partial \bar{c}_y}, m = \frac{\partial u}{\partial \bar{\sigma}}.$$

The function u can be represented as the potential of the external force. From the static fundamental equations 1) and 2), it now follows that

$$\frac{d\bar{r}}{ds}\frac{d\bar{c}}{ds} + \frac{dR}{ds}\frac{d\bar{\theta}}{ds} + \overline{\sigma r}\frac{d\bar{\theta}}{ds} + \frac{du}{ds} = o$$

or $$\frac{d\bar{r}}{ds}\bar{\sigma} + \overline{\sigma r}\,\bar{\omega} + P\bar{\omega}\frac{d\omega}{ds} + \frac{du}{ds} = o.$$

Now putting $$\bar{r} = r_\sigma\bar{\sigma} + r_\nu\bar{\nu},$$

then $$\frac{d\bar{r}}{ds} = \frac{dr_\sigma}{ds}\bar{\sigma} + \frac{dr_\nu}{ds}\bar{\nu} + \omega r_\sigma\bar{\nu} + r_\nu\frac{d\bar{\nu}}{ds}$$

and $$\bar{\sigma}\frac{d\bar{r}}{ds} = \frac{dr_\sigma}{ds} + r_\nu\bar{\sigma}\frac{d\bar{\nu}}{ds}$$

However, $$\frac{d\bar{\nu}}{ds} = -\omega\bar{\sigma},$$

and it follows that $$\bar{\sigma}\frac{d\bar{r}}{ds} = -\omega r_\nu + \frac{dr_\sigma}{ds}, \text{ and } \sigma r = r_\nu\bar{\eta}.$$

From this it is seen that the equation $$\frac{dr_\sigma}{ds} + P\bar{\omega}\frac{d\bar{\omega}}{ds} + \frac{du}{ds} = o.$$

is integrable, so that

3) $$\tfrac{1}{2}P\omega^2 + r_\sigma + u = h^o.$$

This equation shows a certain analogy with the principle of the living force in kinetics.

c) For elastic bodies with no external forces acting, u can be taken as zero. In this case, equation 3) takes the simplified form

3') $$\tfrac{1}{2}P\omega^2 + r_\sigma = h^o$$

The virtual work of bending is $R\delta\theta$. From this the equation

$\frac{d}{ds}(R\delta\theta) = \frac{dR}{ds}\delta\theta + R\frac{d\delta\theta}{ds}$ is formed, or with the use of equation 2),

$$\frac{d}{ds}(R\delta\theta) = -r_\nu\delta\theta + R\delta\omega$$

Denoting the end points of the elastic curve by A and B, the integration along the axis of the wire gives

$$\Big[R\delta\theta\Big]_A^B = \int_A^B (R\delta\omega - r_\nu\delta\theta)\,ds.$$

Setting the virtual distortion equal to zero at the boundaries A and B gives $\int_A^B (R\delta\omega - r_\nu\delta\theta)\, ds = o$, or, since $r_\nu\delta\theta = dr_\sigma$,

$$\delta\int_A^B (\tfrac{1}{2}P\omega^2 - r_\sigma)\, ds = o.$$

By equation 3'), however, $-r_\sigma = \frac{1}{2}P\omega^2 - h^0$. Hence it follows that

$$4) \qquad \delta\Big[\int_A^B P\omega^2\, ds - h^0 \int_A^B ds = o.$$

According to EULER's conception, the integral $\int_A^B P\omega^2\, ds$ is therefore a maximum-minimum with the isoperimetric condition $\int_A^B ds = l$ (constant). From equation 1) follows $\overline{r} = \overline{r^0}$. Therefore $r_\sigma = r_x^\circ \cos\theta + r_y^\circ \sin\theta$, and $r_\nu = -r_x^\circ \sin\theta + r_y^\circ \cos\theta$.

Usually the axes Ox, Oy are so selected that $r_y^\circ = o$.

d) In equation 3) the quantity $\frac{1}{2}P\omega^2 = e$, can be considered as an energy. The sum $u + r_\sigma = u'$ can be considered as a modified potential energy. Setting $e - u' = f$, and defining f as the static LAGRANGIAN Function (in analogy to the kinetics of solid bodies), the static analogue to the LAGRANGIAN kinetic equations has the form $\frac{d}{d\tau}\frac{df}{d\omega} - \frac{df}{d\theta} = o$, and in the present case is identical with the equation $\frac{dR}{ds} + r_\nu + m = o$.

Thereby the analogy of KIRCHHOFF is presented. Further discussions on this analogy are found in LOVE " Elasticity " 2d ed., p. 382, and W. HESS, " Math. Ann." Vol. 25, 1885.

e) EULER gives the method of treating the isoperimetric problem in chapter 5 of " Methodus inveniendi," the German treatment of which, by P. STÄCKEL, is contained in Number 46 of " Ostwald's Klassiker der exakten Wissenschaften."

(K. HEUN).

Note 1, *page* 78.

DANIEL BERNOULLI pointed out the potential force to EULER in a letter dated October 20, 1742 (letter 26 in Vol. 2 of FUSS " Correspondance mathémat. et physique," Petersburg, 1843). He says at the close of this letter : " Since no one is so completely the master of the isoperimetric method (i.e., the calculus of variations, which EULER founded as a special branch of analysis) as you are, you will very easily solve the following problem in which it is required that $\int \frac{ds}{R^2}$ shall be a minimum." DANIEL BERNOULLI knew of EULER's " Method of finding Curves," together with the supplement on elastic curves, before its appearance; for he spoke about these with great interest in his letters to EULER in 1743. See section 63 and note 30.

Note 2, *page* 78.

For the special formulas employed here, compare Chapters II. and V. of EULER's "Methodus inveniendi lineas curvas etc." (Vol. 46 of "Ostwald's Klassiker der exakten Wissenschaften," edited by P. STÄCKEL.)

Note 3, *page* 81.

For this transformation of the co-ordinates, let the following facts be noticed: The new axes are again at right angles; the new x-axis forms an angle ϕ with the old x-axis defined by $\tan\phi = \frac{\gamma}{\beta}$. Putting $P = \frac{dY}{dX}$, this gives $p = \frac{\beta P - \gamma}{\beta + \gamma P}$,

and therefore $1 + p^2 = \frac{(\beta^2 + \gamma^2)(1 + P^2)}{(\beta + \gamma P)^2}$

Substituting this value in EULER's last equation, namely

$$\frac{2\sqrt{a\sqrt{1 + p^2} + \beta p + \gamma}}{(1 + p^2)^{\frac{1}{4}}} = \beta x - \gamma y + \delta, \text{ gives the result}$$

$$\frac{2\sqrt{a\sqrt{1 + P^2} + P\sqrt{\beta^2 + \gamma^2}}}{(1 + P^2)^{\frac{1}{4}}} = X\sqrt{\beta^2 + \gamma^2}$$

Let $\beta_1 = \sqrt{\beta^2 + \gamma^2}$

Introducing lower case letters instead of capital letters gives

$$2\sqrt{a\sqrt{1 + p^2} + \beta_1 p} = \beta_1 x (1 + p^2)^{\frac{1}{4}}$$

Again, writing β for β_1 this last equation reduces to the equation given in the text.

Note 4, *page* 82.

$$\text{Put } n = \gamma,\ x = x_1 + \frac{\beta}{2\gamma},\ m = \frac{1}{a^2}\left(\frac{\beta^2}{4} - a\gamma\right).$$

The quantities a, β, γ, used here are, of course, different from those given at the beginning of this section. This gives

$n^2x^2 - ma^2 = \gamma(a + \beta x_1 + \gamma x_1^2)$, and therefore

$$dy = \frac{dx_1(a + \beta x_1 + \gamma x_1^2)}{\sqrt{a^4 - (a + \beta x_1 + \gamma x_1^2)^2}}$$

Omitting the subscript on x gives the next to the last equation in section 4 of the text.

Note 5, *page* 89.

Here EULER is thinking of NEWTON's famous enumeration of curves of the third order. In the following discussion, AP (Fig. 6) is always the direction of the positive x-axis, and AB the direction of the positive y-axis. AB is also the direction of the external force. In section 5 the direction of the external force is parallel to the *negative* y-axis.

Note 6, *page* 91.

The shape of the curve in the neighborhood of C can be also derived as $u = a\int\sqrt{\frac{t}{c}}$, so that $u^2 = a^2\frac{t}{c}$ represents a parabola. The curve, near C,

is approximately a parabola. If x is made very small in the original equation, then $dy = \frac{(a^2 - c^2)\,dx}{c\sqrt{2a^2 - c^2}}$, and therefore $y = \frac{a^2 - c^2}{c\sqrt{2a^2 - c^2}}\,x$, that is, the curve has the form of a straight line in the neighborhood of A. This also follows from the fact that A is a point of flexion of the curve; for we have

$$\frac{d^2y}{dx^2} = 2x\,\frac{a^4}{\sqrt{(c^2 - x^2)^3\,(2a^2 - c^2 + x^2)^3}},$$ which vanishes for $x=o$.

The curve has flexion points for no other values of x.

Note 7, *page* 92.

The elastic curve has been treated in a few places, although not in detail, in "Methodus inveniendi lineas curvas" by EULER. (Vol. 46 of "Ostwald's Klassiker der exakten Wissenschaften," pp. 110, 111, 127, 131.) In chapter 5, par. 46, EULER demonstrates the important property, that of all the curves of the same length which all pass through the same two points, the elastic curve is that one which, when rotated about an axis, generates the solid of greatest volume. He also mentions there the relation $R = \frac{a^2}{2x}$: the radius of curvature is inversely proportional to the abscissa.

Note 8, *page* 92.

If $\frac{x}{c} = u$, and $\frac{c^2}{c^2 - 2a^2} = k^2$, then $s = \frac{a^2}{\sqrt{2a^2 - c^2}}\int \frac{du}{\sqrt{(1-u^2)\,(1-k^2u^2)}}$

that is, according to the definition of LEGENDRE, s is an elliptic function of the first order. By the same substitution, y is transformed into

$$y = \sqrt{2a^2 - c^2}\int \frac{\sqrt{1-k^2u^2}\,du}{\sqrt{1-u^2}} - \frac{a^2}{\sqrt{2a^2 - c^2}}\int \frac{du}{\sqrt{(1-u^2)\,(1-k^2u^2)}}$$

The first member is an elliptic integral of the second order, and the second member is another such integral of the first order. Therefore the integration for s and y cannot be put into a condensed form.

Note 9, *page* 93.

The notation of the limits of the definite integral has been added for brevity, although they are not found in EULER's work, the notation being first adopted by FOURIER in 1822, in the "Traité analytique de la chaleur."

Note 10, *page* 94.

$y = \frac{a}{\sqrt{2}}\,\text{arc sin}\,\frac{x}{c}$ gives $x = c\,\sin\,\frac{y\sqrt{2}}{a}$. In modern terminology, the curve represented by this equation is called a sinusoidal curve. By the term trochoid is now meant a curtate or prolate cycloid; the sine curve can be considered as a special case of a curtate cycloid.

Note 11, *page* 95.

With the aid of the Legendrian relation $KE' + K'E - KK' = \frac{1}{2}\pi$, the EULER relation $4bf = \pi a^2$ is easily derived. (For the necessary formulas on the

elliptical integrals for this, compare for example, E. PASCAL "Repertorium der höheren Mathematik," German ed. by A. SCHEPP, p. 156.) We have

$$f = \int_0^a \frac{a^2 dx}{\sqrt{(a^2 - x^2)\ (a^2 + x^2)}}.$$

Putting $x = a \cos\phi$ gives

$$f = \frac{a}{\sqrt{2}} \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - \frac{1}{2}\sin^2\phi}} = \frac{a}{\sqrt{2}} K. \quad \text{Further,}$$

$$b = \int_0^a \frac{x^2 dx}{\sqrt{(a^2 - x^2)\ (a^2 + x^2)}} = \int_0^a \frac{dx\sqrt{a^2 + x^2}}{\sqrt{a^2 - x^2}} - \int_0^a \frac{a^2 dx}{\sqrt{(a^2 - x^2)\ (a^2 + x^2)}}.$$

Again, putting $x = a \cos\phi$, then

$$b = a\sqrt{2} \int_0^{\frac{\pi}{2}} d\varphi \sqrt{1 - \tfrac{1}{2}\sin^2\varphi} - \frac{a}{\sqrt{2}} K = \frac{a}{\sqrt{2}} (2E - K).$$

As the formulas show, the complete integrals K and E belong, in this case, to the modulus $k^2 = \frac{1}{2}$, and therefore $1 - k^2 = \frac{1}{2}$ also. Putting $1 - k^2$ for k^2 in E and K transforms them into E' and K'. Here, therefore, $K = K'$, and $E = E'$. The above LEGENDRIAN relation gives as a result the equation: $KE' + K'E - KK' = 2KE - K^2 = K(2E - K) = \frac{1}{2}\pi$. But $bf = \frac{1}{2}a^2K\ (2E - K)$, and therefore $4bf = \pi a^2$. Another proof is found in TODHUNTER, "A History of the Theory of Elasticity," Cambridge, 1886, Vol. 1, p. 36.

Note 12, *page* 96.

In this calculation, the author has made an error. EULER puts $b = 1.1803206 \frac{a}{\sqrt{2}}$, instead of $b = \frac{a}{1.1803206\sqrt{2}}$ (this has been corrected in the text).

It then follows that $\frac{b}{a} = 0.59896$, that is, approximately 0.6. From this follows the formula $f = \frac{5a}{6} \cdot \frac{\pi}{2}$. In EULER's text is given the incorrect value $\frac{b}{a} = 0.834612$.

Note 13, *page* 97.

W. HESS treats the problem of elastic curves as an analogue to the oscillation of a pendulum, and gives a series of figures on the possible forms. (Mathem. Annalen 25, 1885).

The direction of the force in EULER's curves of the fifth class, that is, the perpendicular from A on AP, (Fig. 8), forms, in EULER's work, the angle $90^\circ + 40^\circ\ 41' = 130^\circ\ 41'$ with the curve. Hess gives 129.3°. He took this angle from LEGENDRE's Table of Elliptic Integrals; it consists of finding the value of the modulus k^2 for which $2E - K$ vanishes. H. LINSENBARTH, editor of Volume 175 of "Ostwald's Klassiker der exakten Wissenschaften," says, in Note 25, p. 117 of that work, "As the editor has convinced himself, this gives also in this manner the value 130° 41′, and in addition the verification of the calculation from EULER's equation for v shows that EULER calculated correctly,

since otherwise the value 0.8261 would obtain for v, instead of EULER's value 0.8259."

Note 14, *page* 99.

Here the well known relation $\log_e n = \log_{10} n . \log_e 10$ has been employed. $\log_e 10 = 2.302585$. At the end of the section, $L\ QOM$ is set equal to $2\theta - 90^\circ$. To derive this, let $L\ QOM = \varphi$. Therefore $\tan\varphi = \frac{dy}{dx} = \frac{\frac{1}{2}c^2 - x^2}{x\sqrt{c^2 - x^2}}$. Setting $x = c\cos\theta$ gives $\tan\varphi = \frac{1 - 2\cos^2\theta}{2\sin\theta\ \cos\theta} = -\cot 2\theta$. Therefore $\tan\varphi = \cot(180^\circ - 2\theta) = \tan(2\theta - 90^\circ)$. Hence $\varphi = 2\theta - 90^\circ$.

Note 15, *page* 100.

If c is interchanged with g, the curve remains unchanged. Therefore the shape of the curve near G must be similar to the shape near C, except that the curvature at C is greater than at G. The perpendicular from G to Dd is therefore also a diameter of the curve, as is the perpendicular from C to Dd.

Note 16, *page* 102.

Equation I is the equation at the end of section 5, if x is replaced by t, h by c, and if $mEk^2 = -Pf$. This relation is derived at the end of section 10. If the origin is translated from A to C, then I goes into II. The normal equation (III) of the elastic curve does not change, since in it only dy, and not y is moved when the x-axis is displaced parallel to itself. The origin will then be a suitable point of the straight line AB (Fig. 6). In this figure, the external force acts in the direction AB, as is the case in II at the point C. The change of sign before the dx in II is accounted for by the fact that in figure 3, section 5, the force acts in the direction of the negative y-axis, while in figure 6 it acts in the direction of the positive y-axis. (See note 5.) Therefore, since II and III are referred to the same co-ordinate axes, these equations can be brought into agreement.

Note 17, *page* 104.

In this calculation EULER has made an error (corrected in the text). He used only the first four terms of the series expansion given in the text, and obtained the following value for f:

$f = g - \frac{37g^5}{30c^4}$, which is less than g. From a glance at the figure, it is evident that f is greater than g. EULER seems to have overlooked this fact. The value for h is the same in both cases. The detailed calculation given in the text is not given in the original.

Note 18, *page* 106.

Here EULER refers to the formulas given in his "Methodus inveniendi lineas curvas," Chapter IV, sec. 7, II, p. 132.

Note 19, *page* 108.

This relationship can be proved in detail as follows: Let any straight line CP, (Fig. 3) whose equation is $Ax + By + C_1 = o$, be the axis. Then the moment of the force P at the point M (corresponing to the developments in sec. 5) equals

P.CP. Let the point M have, for the time being, the co-ordinates ξ and η, giving the equation of MP as $A(y-\eta) - B(x-\xi) = o$. If the point C be given the co-ordinates $x = k$, $y = l$, then $Ak + Bl + C_1 = o$, and the length of the perpendicular from C to MP, i.e., CP, is given by $\frac{A(l-\eta) - B(k-\xi)}{\sqrt{A^2 + B^2}}$. The moment $P.CP$ is, if ξ and η be again replaced by x and y, $\frac{P(Al - Bk)}{\sqrt{A^2 + B^2}} + \frac{BP}{\sqrt{A^2 + B^2}}x - \frac{AP}{\sqrt{A^2 + B^2}}y$, corresponding to $\alpha + \beta x - \gamma y$ in the text.

Note 20, *page* 112.

The monograph of JOHN BERNOULLI "De Motu Reptorio" is found in the *Acta Erudit.*, Aug. 1705 (Works I, p. 408.)

Note 21, *page* 113.

EULER himself further exercised his powers on both of these definite integrals, for he says in Vol. 4 of the "Institutiones Calculi Integralis" (Petersburg, 1794) on p. 339: "I recently found by a happy chance, with the aid of a quite unique method, that (see also sec. 54) $\int_o^\infty \frac{d\theta \cos\theta}{\sqrt{\theta}} = \frac{\pi}{2}$, and also $\int_o^\infty \frac{d\theta \sin\theta}{\sqrt{\theta}} = \frac{\pi}{2}$."

The curve analyzed here is the one whose natural equation is $rs = a^2$.

This has been called the Klothide by CESARO. Particulars on this, and figures, which appear also in the above mentioned work by EULER, are found in LORIA "Spezielle algebraische und transzendente Kurven der Ebene," German ed. by F. SCHÜTTE, Leipzig, 1902, p. 458.

Note 22, *page* 114.

Namely, consider the interval from zero to infinity to be divided up into the following parts: 1. from o to π, 2. from π to 2π, 3. from 2π to 3π, etc. Then

$$x = \frac{b}{2}\left[\int_o^\pi \frac{dv \sin v}{\sqrt{v}} + \int_\pi^{2\pi} \frac{dv \sin v}{\sqrt{v}} + \int_{2\pi}^{3\pi} \frac{dv \sin v}{\sqrt{v}} + \ldots\ldots\right].$$

Putting $v = v_1 + \pi$ in the second integral, $v = v_2 + 2\pi$ in the third integral, etc., gives

$$x = \frac{b}{2}\left[\int_o^\pi \frac{dv \sin v}{\sqrt{v}} - \int_o^\pi \frac{dv_1 \sin v_1}{\sqrt{v_1 + \pi}} + \int_o^\pi \frac{dv_2 \sin v_2}{\sqrt{v_2 + 2\pi}} - \ldots\ldots\right].$$

Since the relation between the variables is immaterial in the definite integral, v can be put equal to $v = v_1 = v_2 = v_3 = \ldots$ Similarly the value for y may be obtained.

Note 23, *page* 127.

The Latin text reads "...; indeque auferendo logarithmum sinus totius,..." or "and subtracting from it the logarithm of the entire sine," instead of "and subtracting ten from it" as given in the text. The original is apparently a misprint of some nature. The reading in the text conforms to the method of procedure as given in sec. 31.

Note 24, *page* 128.

Let φ transformed into seconds give the number β. Then, in radians, $\phi = \beta \frac{\pi}{180.60^2}$. Therefore $\log \phi = \log \beta - \log \frac{180.60^2.}{\pi}$ This last logarithm is 5.3144... This explains the appearance of this number in sections 73, 74, and 85. In order to get $\log_c \cot\frac{1}{2}\phi$ EULER used the known formula $\log_c \cot\frac{1}{2}\phi = \frac{1}{M} \log_{10} \cot\frac{1}{2}\phi$. To carry out the multiplication on the right side, he again used common logarithms. $\operatorname{Log}_{10} \frac{1}{M} = 0.362215...$ From this the appearance of this number in the second part of the tabular calculations is understood. At the end of the calculations, EULER used the reguli falsi, as also in sec. 85.

Note 25, *page* 129.

The formula $\frac{A}{B} = \tan\frac{1}{2}\varphi$, which has not as yet been derived, is easily found in the following manner : Adding the first two formulas of sec. 70, which contain only A and B, gives

$$2Ae^{\frac{a}{c}} - 2B\sin\frac{a}{c} + 2A\cos\frac{a}{c} = 0, \text{ or, since } \sin\frac{a}{c} = \cos\varphi, \text{ and}$$

$$\cos\frac{a}{c} = -\sin\varphi, \text{ and } e^{\frac{a}{c}} = \cot\tfrac{1}{2}\varphi,$$

$$\cos\varphi = \frac{A}{B}(\cot\tfrac{1}{2}\varphi - \sin\varphi) = \frac{A}{B}\cot\tfrac{1}{2}\varphi\,(1 - 2\sin^2\tfrac{1}{2}\varphi)$$

or $\cos\varphi = \frac{A}{B}\cot\frac{1}{2}\varphi\ \cos\varphi$. Therefore $\frac{A}{B} = \tan\frac{1}{2}\varphi$.

Note 26, *page* 131.

Here EULER refers to the difference between the oscillations of bodies which are elastic due to stretching—a taut chord, corda elastica, and those which are elastic due to stiffness—an elastic ribbon, lamina elastica.

Note 27, *page* 133.

If the lower tone is C, then the higher tone is slightly lower than G sharp. If C has the frequency N, then G sharp has the frequency $\frac{2}{3}\frac{5}{4} N$ (that is, $6\frac{4}{16}$ instead of $6\frac{4}{15}$ as according to EULER's calculations). See the note on section 79.

Note 28, *page* 133.

Approximately $e^{\frac{5\pi}{2}}(1+\varphi) = \frac{2}{\varphi}$. Therefore $e^{\frac{5\pi}{2}} + \phi\, e^{\frac{5\pi}{2}} = \frac{2}{\phi}$.

By the first approximation in the text, however, $\phi\, e^{\frac{5\pi}{2}} = 2$.

Therefore $\frac{2}{\phi} = e^{\frac{5\pi}{2}} + 2$, i. e., $\phi = \frac{1}{1 + \frac{1}{2}e^{\frac{5\pi}{2}}}$

Letting $v = \frac{1}{a^2}\sqrt{gEk^2\frac{a}{M}}$, then to the various modes of vibration correspond the tones having the frequencies

$1.815^2\,\nu$, $4.69^2\,\nu$, , $\left(\frac{5\pi}{2}\right)^2\nu$, $\left(\frac{7\pi}{2}\right)^2\nu$, ... All these tones have been sought out experimentally by CHLADNI, ("Akustik," Leipzig 1802, pp. 94-103). They are, as he also found for the following cases, in the best agreement with EULER's results.(See note 30.)

Note 29, *page* 136.

The case $\frac{a}{c} = o$, which appears frequently in the following sections, is derived as follows: Since a is not zero, then $c = \infty$. That is, since $c^4 = \frac{Ek^2\,af}{M}$, f must equal ∞. The corresponding isochronous pendulum is infinitely long, and the time for one oscillation is infinite. An infinitely long time is needed to produce an oscillation in the ribbon, i.e., it remains at rest.

Note 30, *page* 137.

Only the case of the oscillation with one knot point, represented in Fig. 23, is excluded in free elastic ribbons, not the others. DANIEL BERNOULLI expressed his astonishment at EULER's error in a letter of Sept. 4, 1743. (Letter 30 in the "Correspondance math. et physique," by FUSS). "These oscillations arise freely, and I have determined various conditions, and have performed a great many beautiful experiments on the position of the knot points and the pitch of the tone, which agree beautifully with the theory. I hesitated whether I should not strike out from the supplement the few words which you say about the matter." In the "Acta Acad. Petrop." 1779, Part 1, p. 103, EULER again takes up the investigation of oscillating ribbons from other points of view, and admits of oscillations with an uneven number of knot points without more ado. There he also treats six classes of oscillations, with consideration of the ends of the ribbon —whether free, fixed on supports, or set in a wall—while here he treats only four classes. LORD RAYLEIGH, in Chapter VIII of the "Theorie des Schalles" (German ed. by Fr. NEESEN, Braunschweig 1879), gives a comprehensive presentation of the transversal oscillations of elastic ribbons. In that very place, for instance, he considers the free oscillations with three knot points. See also STREHLKE, "Poggendorf's Annalen" vol. 27, and A. SEEBECK, "Abhandl. d. Kgl. Sächs. Gesellschaft der Wissensch." 1852.

Note 31, *page* 139.

The deviation in the left table is incorrect. It should read — 2621559, namely 4.73003... — 4.73029... The angle ϕ, then, does lie between 1°-0′-40″ and 1°-1′-0″. Letting $\phi = 1°\text{-}0'\text{-}40'' + n$, the rule of false values gives $\frac{n''}{20''} = \frac{2621559}{53145574 + 2621559}$. This gives n = 0.9402, and hence $\phi = 1°\text{-}0'\text{-}40.9402''$. This value appears, for example, in RAYLEIGH's "Theorie des Schalles," vol. 1, p. 298 ff. Then $\frac{a}{c} = 4.7300408$ for the correct value. Since the mistake has

no influence until the fifth decimal place, the further numerical values are correct to that place. In this and the following section, the correct values are inserted.

Note 32, *page* 141.

EULER had, incorrectly, $e^{\frac{a}{2c}} + e^{-\frac{a}{2c}} = + \frac{2\cos\frac{a}{2c}}{\sqrt{\cos\frac{a}{c}}}$. Since $\frac{a}{c} = 270^0\text{-}0'\text{-}40.94''$, then $\cos\frac{a}{2c} = \cos 135^0\text{-}0'\text{-}20.47''$ is necessarily negative. Therefore the right side of the formula has a positive value, as must be true since $e^{\frac{a}{2c}} + e^{-\frac{a}{2c}}$ is positive. The formula can be derived as follows:

$$e^{\frac{a}{2c}} + e^{-\frac{a}{2c}} = \sqrt{e^{\frac{a}{c}}} + \sqrt{e^{-\frac{a}{c}}} = \frac{(1 - \sin\frac{a}{c}) + \cos\frac{a}{c}}{\sqrt{\cos\frac{a}{c}\left(1 - \sin\frac{a}{c}\right)}}.$$

This value has already been used to calculate $\frac{1}{b}(A\, e^{\frac{x}{c}} + Be^{-\frac{x}{c}})$. Introducing half angles,

$$e^{\frac{a}{2c}} + e^{-\frac{a}{2c}} = \frac{2\cos^2\left(\frac{a}{2c}\right) - 2\cos\frac{a}{2c}\sin\frac{a}{2c}}{\sqrt{\cos\frac{a}{c}}\sqrt{\sin^2\frac{a}{2c} + \cos^2\frac{a}{2c} - 2\sin\frac{a}{2c}\cos\frac{a}{2c}}}$$

$$= \frac{2\cos\frac{a}{2c}\left(\cos\frac{a}{2c} - \sin\frac{a}{2c}\right)}{\sqrt{\cos\frac{a}{c}}\left(\sin\frac{a}{2c} - \cos\frac{a}{2c}\right)} = -\frac{2\cos\frac{a}{2c}}{\sqrt{\cos\frac{a}{c}}}.$$ For the second root in the denominator, $\sin\frac{a}{2c} - \cos\frac{a}{2c}$ is used, since this value is positive.

The formulas of sec. 89 are changed from EULER's text to the correct values derived here. The last formula is correct in the text, and hence the numerical values are correct. $\frac{Cc}{Aa}$ is negative, since in Fig. 22 these ordinates have opposite directions.

Note 33, *page* 145.

It should be noticed about the tone intervals under consideration that: 1) the first interval, $\frac{160}{57}$ is, in the C-major scale, the interval from the base note C to F sharp of the next higher octave (2.78 instead of 2.81); 2) the second interval, 4:1, is that of the base tone C to the C two octaves higher; 3) the third interval, 9:1, reaches from C as the base note to the note D which lies three octaves

higher. In connection with the practical working out of these oscillations, which EULER considered difficult, see CHLADNI, "Akustik," p. 99.

Note 34, *page* 147.

This agreement takes place for all modes of vibration. EULER's differing statement comes from the fact that he rejects, for free ribbons, the oscillations for an uneven number of knots. See note 30.

Note 35, *page* 148.

These experiments, which are very important for technique, have been performed in a most fruitful fashion. However, the formulas given here do not lead to useful results, since they do not take into consideration the cross section of the elastic ribbon. See, for example, KUPFFER, "Recherches expérimentales sur l'élasticité des métaux." St. Pétersbourg, 1860.

Outline of a History of differential Geometry (*)

(II)

6. — Gauss

If the French drew the most consistent economic and political consequences from their revolution, the Germans on the other hand were more profoundly stimulated by the accompanying intellectual upheaval. France had Napoleon, but Germany had Beethoven, Kant and Gauss. The small town philosopher of Königsberg and the small town mathematician of Göttingen represent as truly the essential aspects of the new era as the little corporal himself.

Germany, after the death of Leibniz, had participated only in the construction of new mathematical ideas in so far as its despots had imported genius to adorn their throne. The same causes that bring the revolution to France revivify the German intellect. Its clearest expression in mathematics is Carl Friedrich Gauss (1777-1855).

For us the activity of Gauss is threefold : as inventor of non-Euclidean geometry, as inventor of intrinsic differential geometry, and as a theoretical geodesist. This geodetical work, as the practical one, was at the bottom of all his geometrical discoveries. Most of them were made in the period between 1815 and 1830. From 1821 to '25 Gauss was busy surveying the Kingdom of Hanover.

We shall not discuss Gauss' work on non-Euclidean geometry, though it profoundly affected later differential geometry. It has been done in considerable detail. We shall deal only very

(*) See *Isis*, 19, 92-120.

briefly with his geodetical papers. Our main subject has to be GAUSS' direct intervention into the theory of surfaces.

GAUSS' mathematical investigation of the problem of map projection was written in 1822 under the title " Allgemeine Auflösung der Aufgabe die Theile einer gegebnen Fläche auf einer andren gegebnen Fläche so abzubilden dass die Abbildung dem Abgebildeten in den kleinsten Theilen ähnlich wird." (1) It is the general theory of the conformal representation of arbitrary surfaces, and it represents a continuation of papers by EULER and LAGRANGE. As EULER and LAGRANGE, GAUSS uses " Gaussian coordinates," taking x, y, z, equal to functions of t and u, but where EULER and LAGRANGE represent a surface only on a plane, GAUSS represents two arbitrary surfaces upon each other. The solution, which uses complex functions, is applied to different cases, for instance, that of mapping a rotation ellipsoid on a sphere.

Later GAUSS came back to this subject and sought that conformal representation of the rotation ellipsoid on a sphere which is best for certain purposes. (2)

This work led GAUSS to the general theory of surfaces. A manuscript on this subject dates from 1825. He published a revised paper on this theory in 1827. (3)

These *Disquisitiones generales circa superficies curvas* of 1827 mean an entire departure from the French methods. (4) A new line of approach is found in the investigation of the intrinsic properties of surfaces, depending only on the linear element, and not the properties of the surface as imbedded in Euclidean three-space. Such intrinsic properties are unchanged by bending

(1) Printed in *Astronomische Abh.* 1825. See GAUSS Werke IV, p. 189-216. The paper was a solution to a problem posed by the Academy at Copenhagen in 1822.

(2) C. F. GAUSS, Untersuchungen über höhere Geodäsie, I, II, *Göttingen Abh.* 2 (1844), 3 (1847), GAUSS' Werke IV, p. 259-300, 301-340.

(3) The older ms. has the title : " Neue allgemeine Untersuchungen über die krummen Flächen " (1825), Werke VIII, p. 408 (with remarks of STÄCKEL). The " Disquisitiones " were published in *Commentationes soc. reg. sc. Göttingensis*, 6 (1828), Werke IV, p. 217-258. English translations of both papers by J. C. MOREHEAD and A. M. HILTEBEITEL. Princeton Univers. Library, 1902, p. 126, with bibliography of papers using GAUSS' method, and many notes.

(4) That GAUSS was acquainted with MONGE's contributions to geometry is shown by his favorable review of MONGE's " Géométrie descriptive," *Gött. gel. Anzeiger* 1813, Werke IV, p. 359-360.

the surface without tearing or stretching. GAUSS writes the linear element

$$ds = \sqrt{(Edp^2 + 2Fdpdq + Gdq^2)},$$

where p and q are "Gaussian" curvilinear coordinates on the surface. The main theorem is that the quantity $k = \frac{1}{R_1 R_2}$, a result from EULER's paper of 1760, depends only on E, F, G, and their first and second derivatives. GAUSS proves this "theorema egregium" in more than one way, first by taking the general linear element, and then by assuming $ds^2 = dp^2 + Gdq^2$. This involves a discussion of geodesics, and the discovery of another outstanding theorem, the one dealing with the sum of the angles in a geodesic triangle. This theorem is one of the first examples of so-called differential geometry in the large, and was partly anticipated by the corresponding theorem of LEGENDRE on the sum of the angles of a spherical triangle. (5) GAUSS proves the theorem by a transformation of the surface integral of curvature k, the "curvatura integra," into a line integral. The paper ends with expansions in series of different functions on the surface, in a coordinate system $ds^2 = dp^2 + G\,dq^2$.

These results constitute the essentials of the intrinsic theory of surfaces, linear element, geodesics, Gaussian curvature, curvatura integra, and geodesic triangle. Later investigations have not been able to add many more essential elements; they have shown only that with linear element and Gaussian curvature the theory is complete. One simple conception seems to be lacking, that of geodesic curvature, but in GAUSS' unpublished paper of 1825 this also appears, as "Seitenkrümmung."

GAUSS makes much use of the spherical representations of a surface, an idea he derived from astronomy, and indeed he defines the curvature k as the limit of the quotient of a small area of the surface and the corresponding area of the unit sphere. This raises, perhaps, the question of GAUSS' originality. Spherical representation, geodesics, curvilinear coordinates are all to be found in EULER—why admire GAUSS for their discovery? RODRI-

(5) A. M. LEGENDRE, *Mem. div. sav.* 1787, p. 358; also in the Appendix, Section 5, to his trigonometry. LEGENDRE's theorem was known to GAUSS Disquisitiones, Sec. 27.

GUES even had found the ratio between the surface elements of surface and unit sphere. But, though they were known to EULER, nobody had sensed their importance before GAUSS, and that after fifty years of productivity in France. And the light that GAUSS throws on both old and new conceptions as intrinsic properties of the surface was altogether new.

The invariance of the curvature under bending has been found in papers of GAUSS dating back to 1822. In these papers he uses the linear element of his paper on map projection, $ds^2 = m^2 (dt^2 + du^2)$.

However, nothing in the *Disquisitiones*, as KLEIN remarks, (6) shows that GAUSS kept his boldest ideas to himself. His manuscripts show that he was in possession of the non-Euclidean geometry when writing his paper on surfaces, but he did not hint at it even remotely. But it throws a new and better light upon his theorem on the sum of the angles of a geodesic triangle. It was a theorem enabling the physicist to test the nature of physical space. It was a step in the direction later consistently followed by RIEMANN and HELMHOLTZ. GAUSS indeed measured geodesical triangles, but did not find confirmation of his belief that there might be a perceptible curvature (negative, as GAUSS thought, in agreement with the later theories of LOBATSCHEWSKY and BOLYAI) of space.

Of GAUSS' contribution to notation and nomenclature we mention the symbols E, F, G, D, D', D'' for what we now call the coefficients of the first and second fundamental differential form, and the word " conformal." (6a)

GAUSS became the teacher of the entire learned world, but he created no direct school as MONGE had done. Conditions in Germany were different from those in France. The only exception, so far as differential geometry is concerned, was FERDINAND MINDING (1806-1885) in far-away Dorpat, who filled the

(6) F. KLEIN, Vorlesungen über die Entwicklung d. Mathem. I, p. 16. Here are excellent descriptions of the work of GAUSS.

(6a) In the first paper on higher geodesy, 1844 : " ich werde daher dieselben conforme Abbildungen oder Übertragungen nennen, in dem ich diesem sonst vagen Beiworte eine mathematisch scharf bestimmte Bedeutung beilege," Werke IV, p. 262. The word is, indeed, already used by F. T. SCHUBERT, " De projectione sphaeroidis ellipticae geographica," *Nova Acta Petr.*, p. 130-146, see CANTOR IV, p. 575.

gap left by GAUSS and published a paper on the geodesic curvature (6b). Some years later MINDING published other papers of outstanding importance, all showing a deep understanding of GAUSS' ideas. In them he took up the " problem of MINDING," that is the question of the applicability of surfaces on each other. One of his theorems states that two surfaces of constant curvature are always applicable. (6c) He determined the rotation surfaces of constant positive curvature and established the existence of the helicoids of constant curvature. He also remarked (as EULER had done) that a closed convex surface cannot be bent.

JACOBI also grasped immediately the importance of GAUSS' work in geometry and explained it in his lectures. C. G. J. JACOBI (1804-1851) was at that time at Königsberg. In a paper of 1836 he generalized GAUSS' theorem on geodesic triangles on a surface. (7) But JACOBI's contributions to differential geometry are much more important than this. He beautifully combined GAUSS' ideas with his own. Through his discoveries on Abelian integrals he was able to integrate the geodesic lines on an ellipsoid, which led to hyperelliptic integrals, a counterpart to MONGE's integration of the lines of curvature. (8) In the calculus of variations he established the existence of the conjugate points on the geodesics passing through a point on a surface.

(6b) F. MINDING, Bemerkung über die Abwicklung krummer Linien und Flächen. *Crelle* 6 (1830), p. 159.

(6c) F. MINDING, Wie sich entscheiden lässt ob zwei gegebene krumme Flächen auf einander abwickelbar sind oder nicht. *Crelle* 19 (1839), p. 370. In *Crelle*, from 1838-40, are several papers of MINDING on this subject, the remark on convex surfaces in 18 (1838), p. 365-368.

Through MINDING, his colleague K. E. SENFF (1810-1849) and their pupil, KARL PETERSON (1828-1881), Dorpat figures as a minor center in the history of differential geometry. SENFF found several results later discovered by SERRET. PETERSON, who later became teacher at Moscow, wrote a book " Ueber Curven und Flächen " (1868) with remarkable results, and in an unpublished " Kandidatenschrift " of 1853 found BONNET's theorem of 1866 on a surface being determined by first and second fundamental form and equations equivalent to those of MAINARDI. See P. STÄCKEL, *Bibl. mathematica* (3) 2 (1901), p. 122-132.

(7) C. G. J. JACOBI, Demonstratio et amplificatio nova theorematis Gaussiani de curvatura integra trianguli in data superficie e lineis brevissimis formati, *Crelle* 16 (1836) p. 344-358; see also note 10.

(8) C. G. J. JACOBI, *C. R.* 8 (1839), p. 284; *Crelle* 19 (1837), p. 309, Vorlesungen über Dynamik, p. 116.

On an ellipsoid these conjugate points form a curve with four cusps. (9)

In this work JACOBI touches on differential geometry in the large. Another example is his theorem that the spherical image of the principal normals to a closed continuously curved space curve divides the surface of a sphere into two equal parts, (10) a corollary of his generalization of GAUSS' theorem on geodesic triangles.

JACOBI regularly gave lectures on the theory of curves and surfaces, some of which still exist in manuscript. In KÖENIGSBERGER's book on JACOBI excerpts are given. They show a vivid interest in the production of the French school and GAUSS, and they present, with JACOBI's own work, much of this material in an original way.

An early pupil of GAUSS and JACOBI in Germany was F. JOACHIMSTHAL (1816-1861) to whose work we have to come back. A pupil of JACOBI at Königsberg was HEINRICH FERDINAND SCHERK (1798-1885), later teacher of mathematics at Kiel and Bremen, who found a minimal surface, differing from the two found by MEUSNIER more than fifty years earlier. (11)

One of the reasons for the small number of workers on differential geometry in those days was, of course, the great attraction exercised by the great algebraic geometers, STEINER, MÖBIUS, PLÜCKER, CHASLES, PONCELET, and others. STEINER and MÖBIUS occasionally ventured into infinitesimals, and the result was always interesting. For instance, STEINER found, in his original, thoroughly geometrical, way, the properties of the curvature centroid of closed plane curves (1838), and renewed in classical way the isoperimetrical problem (1841). A brief paper of 1839 deals with involutes of space curves; it is related to JACOBI's paper on space curves of that year. (12) MÖBIUS, in his *Barycentrische Calcül* (1827)

(9) C. G. J. JACOBI, Posthumous paper: Über die Curve, welche alle von einem Punkte ausgehenden geodätischen Linien eines Rotationsellipsoides berührt. Ges. Werke VII, p. 72-80.

(10) C. G. J. JACOBI, Über einige merkwürdige Curventheoreme, *Astron. Nachr.* 20 (1842), p. 115-120. Ges. Werke VII, p. 34-39.

(11) H. F. SCHERK, Bemerkungen über die kleinste Fläche innerhalb gegebener Grenzen. *Crelle* 13, 1835, p. 185; earlier in a Latin paper in *Acta Soc. Jablonovianae* 4 (1830).

(12) J. STEINER, Einfache Beweise der isoperimetrischen Hauptsätze, *Crelle*

has three chapters on curves and surfaces, treated for the first time in an invariant symbolism. His work leads to DUPIN's results on the indicatrix, and he claims to be an independent discoverer of its properties; it was only after he had his results, he writes, that he discovered them in DUPIN's *Développements*. (13)

Accordingly, not only in Germany was there little immediate response to GAUSS' work, but also in France, where, moreover, the MONGE tradition was still very much alive. We may even feel a certain polemical note in a paper written by SOPHIE GERMAIN, (at that time already fifty-five), in *Crelle* 1831, recommending the " mean curvature " $\frac{1}{2}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)$ as the measure of curvature of a surface instead of GAUSS' $\frac{1}{R_1 R_2}$. The mean curvature is indeed of importance in elasticity, a field in which SOPHIE GERMAIN had won her laurels.

7. — *The French School of the Forties.*

The new generation in France considered GAUSS and JACOBI its leaders. GALOIS, in his last letter to a friend, expressed it in this way : " Tu prieras publiquement JACOBI ou GAUSS de donner leur avis non sur la vérité, mais sur l'importance des théorèmes." But there was also a second influence, working in France, namely the development of physics in general and mathematical physics in particular, — a direct result of the tremendous changes in the European industrial system.

During the Restoration this development, begun during the Empire, reached with FRESNEL and AMPÈRE a point where it veritably revolutionized scientific thought. Optics, electricity, elasticity, the theory of heat, and astronomy were affected. MALUS and CHLADLI were among the first venturing into these unknown regions; then followed POISSON, FOURIER, FRESNEL, ARAGO,

18 (1836), p. 281-296, Ges. Werke II, p. 75-91. Von dem Krümmungsschwerpuncte ebener Curven. *Crelle* 21 (1838), p. 33-63, 101-133, Ges. Werke II, p. 97-159. Ueber einige allgemeine Eigenschaften der Curven von doppelter Krümmung. *Monatsber. Akad. Berlin*, 1839, p. 76-80, Ges. Werke II, p. 163-165.

(13) A. F. MÖBIUS, Ges. Werke I, p. 1-388. Differential geometry in Ch. VI. VII, VIII. See Vorrede p. 8.

AMPÈRE, SADI CARNOT, CLAPEYRON, and many others. With NAVIER and SOPHIE GERMAIN, DE SAINT VENANT, and LAMÉ, the field of theoretical elasticity is opened, — an event which was to exert considerable influence on the application of analysis to geometry.

For our purpose the most important figure of this group is GABRIEL LAMÉ (1795-1870), a pupil of the Ecole Polytechnique, who was sent, in the company of CLAPEYRON, to Russia to assist in the improvement of methods of technical instruction. After the July revolution of 1830, he became professor at Paris. This engineering work in the development of new railroads exhibits clearly the close relationship which characterized the progress of natural science and the rapidly growing French capitalism of his time; compare, for instance, also the work of CRELLE on Prussian railroads. In asking for solutions of the equations of heat and elasticity in other than rectangular solids LAMÉ developed a theory of rectangular curvilinear coordinates, in which the line element of space is written

$$ds^2 = H^2\,d\rho^2 + H_1^2\,d\rho_1^2 + H_2^2\,d\rho_2^2$$

and where the quantity

$$\sqrt{\left(\frac{dF}{dx}\right)^2 + \left(\frac{dF}{dy}\right)^2 + \left(\frac{dF}{dz}\right)^2}$$

is what LAMÉ called the " paramètre différentiel du premier ordre " of the function $F\,(x, y, z)$. There are also " paramètres différentiels du second ordre "

$$\frac{d^2F}{dx^2} + \frac{d^2F}{dy^2} + \frac{d^2F}{dz^2}\,.$$

LAMÉ's six partial differential equations of the second order connecting the H form the basis of his investigations, which carry him deep into interesting analytical developments.

Here we find not only " coordonnées curvilignes " in space, but also invariants under rotations and translations. This invariance is indicated by the introduction of the symbols $\Delta_1 F$ and $\Delta_2 F$ for first and second differential parameters of $F\,(x, y, z)$. A single infinity of surfaces $F = \text{const}$, for which $\Delta_1 F = 0$, is called " isotherme," the function F a " paramètre thermométrique " of the family of surfaces.

LAMÉ introduced these notions in a series of publications, the first of which, in 1837, on isothermic surfaces, received immediate

attention. It was followed in 1840 by a paper on curvilinear coordinates. (14) After many years of teaching he collected the fruits of his research in a series of textbooks, which are, even now, very readable. (15)

Another mathematician, who entered the field of differential geometry through his work in applied mathematics, was A. J. C. BARRÉ DE SAINT VENANT (1797-1886), well known as an early contributor to the theory of elasticity. He wrote, in 1846, a paper which supplied the impetus to the final completion of the elementary theory of space curves. (16)

It was under such teachers that the younger mathematicians developed. The leading figure in pure mathematics became JOSEPH LIOUVILLE (1809-1882).

LIOUVILLE's fame is the result, on the one hand, of his diversified mathematical investigations, and on the other, of his success as the founder and editor of the " Journal de mathématiques pures et appliquées." This Journal, first published in 1836, has published a great number of papers on geometry; its parallel in Germany was CRELLE's " Journal " of the same title. CRELLE and LIOUVILLE's Journals constituted the principal agencies for the progress of differential geometry in the middle part of the 19th Century, and as such their importance is not merely of the past.

Among the younger contributors to LIOUVILLE's Journal were J. A. SERRET (1819-1885), V. PUISEUX (1820-1883), O. BONNET (1819-1892), and J. BERTRAND (1822-1900), all of whom became professors at Paris, together with F. FRENET (1816-1868), professor at Lyons, and the Belgian E. CATALAN (1819-1894), who studied at the Ecole Polytechnique and became professor at Liége. Their work is based not only on that of LAMÉ and MONGE but also of GAUSS and JACOBI. We shall try to summarize, without going

(14) G. LAMÉ, Mémoire sur les surfaces isothermes, *Journ. de math.* (10)2 (1837), p. 149; Mémoire sur les coordonnées curvilignes, *ib.* 5 (1840), p. 313-347.

(15) G. LAMÉ, Leçons sur la théorie mathématique de l'électricité (1852); Leçons sur les fonctions inverses des transcendantes et les surfaces isothermes (1857); Leçons sur les coordonnées curvilignes et leur diverses applications (1859); Leçons sur la théorie analytique de la chaleur (1861).

(16) SAINT-VENANT, Mémoire sur les lignes courbes non planes. *Journ. Ec. Polyt.* cah. 30 (1846), p. 1-76.

too far into an estimation of individual merits, the results of this school, which flourished from 1840 to 1850.

1) *Work on Gauss' theorems.* In many papers and by various methods the theorem of GAUSS on the invariance of curvature and the sum of the angles of a geodesic triangle is proved again (17). It is in this work that modern nomenclature evolves more and more, as, R_1 and R_2, $ds^2 = Edu^2 + 2Fdudv + Gdv^2$, etc. BONNET names the " courbure géodésique," (18) and derives the " formula of GAUSS-BONNET." LIOUVILLE introduces as a counterpart to GAUSS' ds^2 in geodesic polar coordinates the $ds^2 = \lambda (du^2 + dv^2)$, already studied by GAUSS in his paper on map projections, and found in LAMÉ's investigations on heat (isothermal element). This leads LIOUVILLE to conformal representation and to " LIOUVILLE's surfaces," for which the geodesics can be found by quadratures. JACOBI's determination of the geodesics on an ellipsoid is a special case. (19) LIOUVILLE also determined all rotation surfaces of constant curvature, completing MINDING's results.

2) *Special problems in the Monge tradition.* MONGE's tradition remained entirely alive. As an example of work accomplished under its influence we mention several studies on surfaces with lines of curvature, plane or spherical. JOACHIMSTHAL, the German geometer, first at Halle, later at Breslau, shared in these investigations, (20) and found, among other results, the " theorem of JOACHIMSTHAL " on plane lines of curvature. In this connection we may also mention ABEL TRANSON (1805-1876), who continued the work of AMPÈRE and CARNOT and who belongs to the early students of affine differential geometry.

3) *The greater prominence of the notion of invariance.* The frequent use of curvilinear coordinates emphasizes GAUSS'

(17) E.g. J. LIOUVILLE, Sur un theorème de M. GAUSS concernant le produit de deux rayons de courbure principaux en chaque point d'une surface. *Journ. de Math.* 12 (1847), p. 291-304.

(18) O. BONNET, Mémoire sur la théorie générale des surfaces. *Journ. Ec. Polyt.* 19, cah. 32 (1848), p. 1-146.

(19) J. LIOUVILLE, Sur quelques cas particuliers où les équations du mouvement d'un point matériel peuvent s'intégrer, *Journ. de Math.* 11 (1846), p. 315. De la ligne géodésique sur un ellipsoide quelconque, *Journ. de Math.* 9 (1844), p. 401, also appendix of the fifth ed. of MONGE's " Applications."

(20) F. JOACHIMSTHAL, Demonstrationes theorematum ad superficies curvas spectantium, *Crelle* 30 (1846), p. 347-350.

standpoint on the invariance of quantities under coordinate transformations. LAMÉ's differential parameters $\Delta_1 F$ and $\Delta_2 F$ are a first effort to create a special notation for differential invariants on the surface. This development is contemporary with that in algebraic invariants, a field in which CAYLEY and ARONHOLD had already started their investigations. These geometers, however, did not arrive at the relationship between the GAUSS coordinates on a surface and the LAMÉ coordinates in space. This could be done only after RIEMANN's fundamental work on the nature of space.

4) *Development of the general theory of space curves.* The theory of space curves lacked both elegance and easy access. DE SAINT VENANT, who wrote an extensive study on space curves in 1846, (16) in which he collected the available material, added several new theorems, introduced the word " binormale," and gave a good historical review, concluding his paper with several pages of formulas. This approach could be abandoned when F. FRENET, in his Toulouse dissertation of 1847, found the key to an easy control of the computations in this field. (21) His " FRENET formulas " are the result of a belief that we should not only find the derivatives with respect to the arc of the direction cosines of the tangent, but also those of the principal normal (rectifying plane) and the binormal (osculating plane). A few years later J. A. SERRET, (22) ignorant of FRENET's dissertation, arrived at the same formulas; this led FRENET to a renewed publication of his results in the same periodical, *Liouville*'s Journal. Their importance was not generally recognized very soon. The textbook of F. JOACHIMSTHAL on differential geometry, which was the result of lectures at Breslau during the winter term of 1856-57, does not give them, (23) not even the third edition of 1890, edited by L. NATANI. Neither do they appear in a book on space

(21) F. FRENET, Sur les courbes à double courbure. Thèse Toulouse, 1847. Abstract in *Journ. de Math.* 17 (1852), p. 437.

(22) J. A. SERRET, Sur quelques formules relatives à la théorie des courbes à double courbure. *Journ. de Math.* 16 (1851), p. 193.

(23) F. JOACHIMSTHAL, Anwendung der Differential- und Integralrechnung auf die allgemeine Theorie der Flächen und der Linien doppelter Krümmung. Leipzig 1872. This excellent introduction, one of the first to appear in Germany, gives a symposium of the results of the French school and those of GAUSS and JACOBI.

curves, written in 1860 by another SERRET (PAUL) (1827-1898), a professor at the Université Catholique at Paris, who gave an interesting exposition of the whole theory as it was known in 1860, with special applications to spherical curves. (24)

5) *Solution of special problems on space curves.* PUISEUX proved that the ordinary helix is the only curve for which curvature and torsion are constant. (25) Then BERTRAND showed that the helix on a general cylinder is the only curve for which the ratio of curvature and torsion is a constant. SERRET found the equation of the curves of constant curvature and that of curves of constant torsion, and BERTRAND discovered the " BERTRAND curves," curves for which a linear relation exists between curvature and torsion. After SERRET had published his formulas, it was recognized that these results could be obtained much more easily from the FRENET formulas. PAUL SERRET's book of 1860 has already been mentioned.

6) *Original results.* It would be entirely wrong to think of these mathematicians only as pupils of great masters. As an example of a new and surprising result we mention LIOUVILLE's theorem, that conformal transformations in space are only inversions, similarity and congruency transformations. LIOUVILLE found this theorem in an investigation of the conformal representation of surfaces; he published it, together with many other results of his own and his associates, in the fifth edition of MONGE's *Applications*. LIOUVILLE also added a reprint of GAUSS' paper and as a result we get from this fifth edition a fair idea of the progress of differential geometry to the fifties of the nineteenth century. (26)

---

(24) PAUL SERRET, Théorie nouvelle géométrique et mécanique des lignes à double courbure, 1860; also Thèse Paris, 1859, without some notes. The book also contains the theory of curves on surfaces, and has several new results, as e.g. the theorem that asymptotic lines of a ruled surface are cut by the generators in 4 points of constant anharmonic ratio (p. 169).

(25) V. PUISEUX, Problème de géométrie. *Journ. de Math.* 7 (1842), p. 65.

(26) A textbook showing the regular class programs of those days in France is C. F. A. LEROY, Analyse appliquée à la Géométrie des trois dimensions. Paris, Mallet Bachelier, 4th ed. 1854, 408 p., of which almost one half is devoted to differential geometry. It is almost exclusively based on the texts of MONGE and DUPIN, influenced by the presentation of these texts by CAUCHY and POISSON (*Journ. Ec. Pol.* XXI). The word " torsion " is used here (p. 292, 299). There is no influence of GAUSS.

8. — RIEMANN

Outside of France there was more interest than initiative. In England the interest in differential geometry could only be awakened after the fossilized epigones of NEWTON had been replaced by younger men, acquainted with LEIBNIZ's methods. This renaissance was initiated by CHARLES BABBAGE and his friends in 1811. But scarcely any differential geometry was investigated till the publication of the *Cambridge Mathematical Journal* (1837), which soon became *The Cambridge and Dublin Mathematical Journal* (1846), in which geometry and analysis were cultivated in the continental style. Differential geometry, however, appeared mainly as an aspect of the theory of conic sections and quadrics. Among the authors in this journal we mention D. F. GREGORY, W. WALTON, A. CAYLEY, and W. THOMSON. GREGORY (1813-1844), who died young, left a well written *Treatise on the application of analysis to solid geometry*, edited by WALTON (1845). It shows the French influence even in its terminology. The importance of GREGORY is shown by the fact that his friends published his collected works. The centers of this new life were at Cambridge and Dublin, and at Dublin we find the leading figure of W. R. HAMILTON (1805-1865), who published between 1828 and 1837 his four papers on systems of rays, which carry on the work of MONGE and MALUS and lead to his fundamental theorems on partial differential equations. (27) He also presented curve and surface theory in his quaternion lectures, published in 1853 and 1866. (28) They contain a complete theory of curves and surfaces, with many new results. He studies, for instance, space curves up to elements of the fifth order with respect to the arc-length. To his circle belong JOHN THOMAS GRAVES (1806-1870), a professor of law at London, who studied mathematics at Dublin, and the twin brothers MICHAEL (1817-1882) and WILLIAM ROBERTS (1817-1883), professors of mathematics at Dublin, who contributed to differential geometry. In the theory of confocal curves on an

(27) W. R. HAMILTON, On systems of rays, *Trans. Roy. Irish Academy* 1828; Supplements I, II, ib. 1833; III, ib. 1837.

(28) W. R. HAMILTON, Lectures on Quaternions, 1853; Elements of Quaternions 1866.

ellipsoid a theorem bears GRAVES' name. From this Dublin center come also the famous textbooks of the theologian-mathematician GEORGE SALMON (1819-1904), who deals to a considerable extent with the application of analysis to geometry, as in *Higher plane curves* (1852) and *Analytic geometry of three dimensions* (1862). England, though responsible for little creative work in our field, had, for a time, the better of the textbooks. In the forties we find in Germany only the work of F. JOACHIMSTHAL, an investigator who understood both the French and the German masters. In Italy differential geometry found excellent teachers in BORDONI and CHELINI, but it was their pupils who were to contribute first class results. Even in France little work of outstanding value is produced in the fifties. (29) A new impulse had to come.

This new impulse comes again from Germany, and again from Göttingen. In 1854, BERNARD RIEMANN (1826-1866) addresses an audience in the little town as part of the requirements for a " Dozentur." He talks " on the hypotheses which serve as foundation to geometry." GAUSS was in his audience. The paper was not printed till 1867, (30) but its publication set in motion an influence which has lasted till the present day.

The actual mathematical facts in RIEMANN's address can be stated in a few words. He develops the conception of a general n-dimensional manifold in the sense of the analysis situs and then introduces into it a quadratic linear element ds^2. He describes how the curvature of such a manifold can be measured, and specializes manifolds of constant curvature. In this way both Euclidean geometry and the non-Euclidean geometry of BOLYAY are obtained as special cases of a general " Riemannian " geometry. As these two cases are represented by a zero and negative constant curvature, the question arises as to a manifold of positive constant curvature, the " Riemannian" non-Euclidean spaces. The results of GAUSS on intrinsic properties of surfaces covered RIEMANN's special case $n=2$.

But the meaning of RIEMANN's work lies far deeper than the

(29) O. BONNET is an outstanding exception.

(30) Über die Hypothesen, welche der Geometrie zugrunde liegen, *Gött. Abh.* 13 (1867), Werke No. XIII, new edition by H. WEYL, Berlin, SPRINGER, 2 Aufl., 1921.

purely mathematical structure which resulted from it. We best see it perhaps when we compare MONGE and RIEMANN. MONGE studies surfaces in ordinary space. This space is for him a given entity, about which no discussion can exist. It is given and the task of the geometer is the same as that of the carpenter: he takes bodies in that space and works with them. There is a perfect separation of surface and space, because there is such a separation between rigid body and space. MONGE in this respect is the representative of the metaphysical materialism of the eighteenth century with its sharply defined conceptions.

RIEMANN does not only ask how rigid bodies behave in space, but also how space is affected by rigid bodies. Space is not the undiscussed given entity of MONGE, but rather an entity subject to the same scientific investigation as are the bodies themselves. Starting with RIEMANN, dialectical materialism begins to replace metaphysical materialism in the foundations of geometry. Space works on bodies, bodies influence space. There may be in the first " Anschauung " an undefined conception of space, but experience alone projects into it the properties which make it the space of our geometry and of our physics, the projective relations, the metrical relations and even the relations of analysis situs. From the abstract space of MONGE which relates bodies at finite distance from each other, we pass to the " field " theory of space as RIEMANN gives it. FARADAY and MAXWELL took the same step for the field of electricity (as WEYL remarks) (31). Also in other branches of science similar steps were taken at the same time, opening the passage from the metaphysical classification of eighteenth-century science to modern conceptions. The same dialectical methods characterize LYELL's work in geology, DARWIN's in biology, MARX' in sociology. From RIEMANN to EINSTEIN is but one step—a step however that had to wait sixty years. RIEMANN actually suggested, at the end of his paper, the possibility of determining the metric of space by the physical masses. This is the more striking as an example of almost superhuman divination, as RIEMANN's actual research in electricity and gravitation followed the classical lines of his day.

RIEMANN wrote but few formulas in his address. He had a

(31) See note (3), " Vorwort des Herausgebers."

chance to work out some of his ideas mathematically in a paper of 1861 on the distribution of electricity on cylinders. (32) Here he had to study the question of bringing the partial differential equation

$$\frac{\delta}{\delta x_1}\left(a_{11}\frac{\delta u}{\delta x_1}+a_{12}\frac{\delta u}{\delta x_2}+a_{13}\frac{\delta u}{\delta x_3}\right)+$$
$$\frac{\delta}{\delta x_2}\left(a_{21}\frac{\delta u}{\delta x_1}+a_{22}\frac{\delta u}{\delta x_2}+a_{23}\frac{\delta u}{\delta x_3}\right)+$$
$$\frac{\delta}{\delta x_3}\left(a_{31}\frac{\delta u}{\delta x_1}+a_{32}\frac{\delta u}{\delta x_2}+a_{33}\frac{\delta u}{\delta x_3}\right)=h\frac{\delta u}{\delta t}$$

a_{ik} functions of the x, into the simplest form. For a general number n of x_i this problem is equivalent to that of transforming the quadratic differential form $\Sigma\, a_{ii'}\, dx_i\, dx_{i'}$. RIEMANN finds the necessary and sufficient conditions that this form can be reduced to a sum of n squares in the form of the vanishing of the four index symbols.

$$(ii', i''i''') = 0.$$

Here, therefore, we have the principal concomitant of Riemannian geometry.

It is, however, underestimating RIEMANN's influence on differential geometry by referring only to his work on the space problem. His work on function theory opened new ways of attacking geometrical problems, as is shown, for instance, by the later investigations of SCHWARZ on minimal surfaces. RIEMANN's introduction of the connectivity of a manifold plays an important role in later differential geometry in the large.

Another study on n-dimensional geometry, from an entirely different point of view, appeared at the same time, or better, reappeared. That was GRASSMANN's *Ausdehnungslehre*. HERMANN GRASSMANN (1809-1877), who was a high school teacher (Gymnasiallehrer) at Stettin, had written a first edition of this work in 1844, which remained almost entirely unknown. He, therefore, rewrote it entirely, and published it again in 1862. (33)

---

(32) Commentatio mathematica qua respondere tentatur quaestioni ab Ill. Academia Parisiensi propositae, 1861, Werke, p. 391-423, only published posthumously.

(33) H. GRASSMANN, Die lineale Ausdehnungslehre. Ein neuer Zweig der Mathematik, 1844. Ges. Schriften I (Leipzig, Teubner), p. 1-139. Die Ausdehnungslehre 1862. Ges. Schriften II, p. 1-511.

Here we find an admirable treatment of the elementary Euclidean (and affine) n-dimensional geometry in the form of a (far from elementary) formal calculus of points and vectors, as " Ausdehnungsgrössen," a consistent direct calculus invariant under the affine group and partly under the rotational group. Not only vectors are introduced, but also what we now call tensors. This work has not only bearing on differential geometry in so far as it inspired other geometers to extend their investigations to more dimensions, but also through the direct application GRASSMANN himself made of his apparatus to the problem of PFAFF, through which this problem first took geometrical form.

RIEMANN certainly is the originator of more-dimensional differential geometry, GRASSMANN however of the symbolical methods introduced for the study of this geometry. Here GRASSMANN's work is supplemented by that of HAMILTON on quaternions. HAMILTON also introduces a symbolical method, of which the vector and scalar product, and the nablaoperator, have been utilized in the modern treatment of differential geometry. The invariant character of differential geometry, clearly expressed by GAUSS and the LIOUVILLE school, has its formal expression here, as is immediately seen by the easy way in which LAMÉ's differential parameters fit into the HAMILTON scheme. HAMILTON, however, has no " Ausdehnungslehre."

RIEMANN and GRASSMANN together form a remarkable example of the ways of development of human knowledge, where one mind, essentially dialectical in nature, hews down barriers and discovers new relationships, and the other, operating more formally, builds up new symbolisms to control the field again through rigid, even frozen, methods. This reveals again the greatness of LEIBNIZ, who, in this respect at any rate, combined the merits of RIEMANN and GRASSMANN.

9. — *The beginning of modern times.*

From now on the development of differential geometry does not follow one main line. In the second part of the nineteenth century different centers and different schools arise, in which the influence of their leaders extends far beyond their immediate location. Three sources of creative study draw immediate atten-

tion, though many different nations participate. These three sources are France, Germany and Italy. Their paramount positions did not arise accidentally. France with its long established traditions was the oldest center and continued uninterruptedly its economic development under Bonapartism, and later under the Third Republic. In Germany and Italy we see, at the same time, the birth of the national state, completed in 1870, and, as a result, an enthusiastic expansion of the capitalist system. The creative influence of this development on mathematical thought is clearly seen in Italy.

Here the national revival was known as the Risorgimento. After a series of wars against Austria and its own despots, the Italian people established the kingdom of Italy.

For a long time there had been much interest in differential geometry, mainly in the North, at the University of Pavia, where it was especially fostered . Here A. M. BORDONI (1789-1860) taught for many years, and published, beginning in 1821, papers on the applications of analysis to geometry, papers which show the influence of GAUSS and the LIOUVILLE school. But his important qualities lie mainly in his power to create interest. As colleague he had G. MAINARDI (1800-1879), as pupils or colleagues, D. CODAZZI (1824-1875), F. BRIOSCHI (1824-1897), L. CREMONA (1830-1903), F. CASORATI (1835-1890), and above all, E. BELTRAMI (1835-1900). Another early geometer of influence was the priest D. CHELINI (1802-1878), who was professor of mathematics and physics at different places, and who stimulated interest through his publications (commencing 1845) and his teaching.

Productive activity rises to a higher level in the fifties, when MAINARDI publishes a paper (34) in which he inquires as to what relations must exist between the six functions E, F, G, D, D', D'' introduced by GAUSS into the theory of surfaces. It is clear that there must be such relations, because the coordinate representation of a surface shows that three functions determine the surface. GAUSS found one relation, and the task is to find others. This leads MAINARDI to four equations, in which we now

(34) G. MAINARDI, Su la teoria generale delle superficie, *Giorn. Istit. Lombardo* 9 (1856), p. 385-398.

recognize the two " Mainardi-Codazzi " equations. At the same time the younger generation begins to publish, BRIOSCHI in 1852, CREMONA in 1855, and CODAZZI in 1858.

All these younger mathematicians were not only scholars, but also organizers and prominent in political activities. They were, accordingly, in a position to influence not only their pupils, but also the instruction of the whole country and build up, in a relatively short time, remarkable scientific enthusiasm. In geometry, Italy became one of the leading countries.

In the sixties France, Germany and Italy begin to participate equally in the development of differential geometry. A problem set by the Paris Academy acted as a stimulus : to study the applicability of surfaces upon each other (1860). This problem, a consequence of GAUSS' theory, had so far commanded the attention of MINDING and BONNET in France, who had framed the problem as that of recognizing when two given surfaces are applicable, that is, when they have the same ds^2. Answers to the Paris question came from BOUR and BONNET in France and from CODAZZI in Italy; WEINGARTEN in Germany wrote a related paper. Here we find the problem as that of finding all surfaces which are applicable to a given surface. EDOUARD BOUR (1832-1866), after whom this problem is called, and who got the first prize, was a young mathematician at Paris, who became professor at the Ecole Polytechnique ; his paper, in the words of LIOUVILLE, " peut être pris comme un beau mémoire de LAGRANGE." (35) BONNET, whose work covers the whole period in which differential geometry took its modern shape, wrote more than one important paper during this period; in 1867 he reached the fundamental result that a surface is perfectly determined, but for its position in space, by its first and second fundamental form, if these forms are related by CODAZZI's equations. (36) CODAZZI's contribution to surface theory (37) was followed by other papers in which

(35) E. BOUR, Théorie de la déformation des surfaces. *Journ. Ec. Polyt.*, 22, cah. 39 (1862), p. 1-148.

(36) O. BONNET, Mémoire sur la théorie des surfaces applicables sur une surface donnée, *Journ. Ec. Polyt.*, 24 (cah. 41), 1865, p. 209-230; 25 (cah. 42, 1867), p. 1. See on BONNET the necrology by P. APPELL. *Comptes rendus* 117 (1893) p. 1014-1024.

(37) D. CODAZZI, Mémoire relatif à l'application des surfaces les unes sur les autres. *Mém. prés. par div. sav.* 27 (1882), Nr. 6; it was written in 1859.

he went deeper into the nature of certain equations used by him in his Paris answer, and he showed how they led to the three "Mainardi-Codazzi equations." He did not know that MAINARDI had reached similar results before. The problem of the applicability of surfaces leads naturally to these equations. BOUR gave them, but only for geodesic polar coordinates.

JULIUS WEINGARTEN (1836-1910), from 1873 to 1903 professor at the Technische Hochschule of Berlin-Charlottenburg, developed in his early papers the theory of the so-called W-surfaces, for which a relation exists between the principal radii of curvature. As the problem of finding all surfaces isometrical to a given rotation surface can be reduced to that of finding all W-surfaces of the same class, WEINGARTEN gave an example of a complete set of isometrical surfaces that can be found only by elimination and quadrature. Before that time only the developable surfaces were known. All the later work of WEINGARTEN is related to this problem of the applicability of surfaces. (38)

The important papers of this period are all of the type characterized by the names LAMÉ and GAUSS, and consciously or unconsciously reflecting the ideas of RIEMANN. The BOUR-MINDING type of problem set by GAUSS is indeed an investigation into the nature of two-dimensional RIEMANN manifolds; the same is true of the BONNET problem on the congruency of surfaces, and the investigations of THEODORE MOUTARD (1827-1901) on infinitesimal isometry. Then we have the fundamental papers of BELTRAMI, who, between the years 1864 and 1868, starting from LAMÉ, led directly into the problems left open by RIEMANN. Finally we have papers immediately due to RIEMANN, as HELMHOLTZ' inquiry into the foundations of geometry, and CHRISTOFFEL and LIPSCHITZ' study of the transformation of quadratic forms.

Several German mathematicians whose main line of work lies

(38) D. CODAZZI, Sulle coordinate curvilinee d'una superficie e dello spazio. I, II, III, *Annali di Matem.* (2) 1 (1867-68), p. 310, 2 (1868-69), p. 101-119, 269-287, "Codazzi-equations", p. 273-274. See on this subject R. v. LILIENTHAL, *Encykl. d. math. Wiss.* III, 3, p. 159.

J. WEINGARTEN, Über eine Klasse aufeinander abwickelbarer Flächen, *Crelle* 59 (1861), p. 382.

See also J. WEINGARTEN, Ueber die Oberflächen, für welche einer der beiden Hauptkrümmungshalbmesser eine Function des andern ist, *Crelle* 62, p. 164; *Crelle* 59, p. 382.

outside this field contribute in these years important papers to differential geometry. KUMMER writes on line congruences, WEIERSTRASS on minimal surfaces, and HELMHOLTZ on the axiomatics of geometry.

EUGENIO BELTRAMI is the most brilliant representative of the group of Italian geometers who were trained during the Risorgimento. He was, from 1862 on, professor at Bologna, Pisa, Bologna, Rome (1873-1877), Pavia, and Rome (1891-1900). One of his first papers was a translation of a work of GAUSS. During his time at Pisa he often met RIEMANN, and his work shows the deep influence of both GAUSS and RIEMANN. In an astonishing tempo he published his fundamental research on differential geometry; in 1864, he produced his extension of LAMÉ's differential parameters to curvilinear coordinates with references to the applicability problem; in 1865, several beautiful theorems on the bending of surfaces; in 1867, the generalization of complex functions to surfaces; and in 1868, his theorems on non-Euclidean spaces, in which he happily combined RIEMANN's ideas with the older notions of GAUSS and LOBATCHEVSKI. Here he showed that there is a representation of non-Euclidean geometry on the pseudosphere, so that there is no possibility of a contradiction in non-Euclidean geometry, and he proved that in manifolds of constant RIEMANN curvature the geodesics can be expressed by linear equations. (39) SCHLÄFLI (1814-1895), from 1852-1891 professor at Bern, then proved the inverse theorem. (40) This brought RIEMANN's ideas definitively out of the realm of speculation into the main body of mathematical investigation, and it resulted in the combination of the ideas of LAMÉ, GAUSS and RIEMANN into one solid theory.

The theory of quadratic forms of n variables was taken up independently by CHRISTOFFEL and LIPSCHITZ in papers published in 1869 in the same periodical, CRELLE's " Journal ". ERWIN

(39) Some of these papers are E. BELTRAMI : Ricerche di analisi applicata alla geometria, *Giorn. di mat.* 2 (1864), 3 (1865); Sulla teoria generale delle superficie, *Atti Ist. Veneto* (215 (1860); Delle variabili complesse sopra una superficie qualunque, *Ann. di Mat.* (2) 1 (1867); Teoria fondamentale degli spazi di curvatura constante, *Ann. di Mat.* (2) 2 (1868-69); Saggio di interpretazione della geometria non-euclidea, *Giorn. di Mat.* 4 (1868). All these papers in his " Opere," I, II.

(40) L. SCHLÄFLI, Nota alla memoria del sig. BELTRAMI. *Ann. di mat.* (2) 5 (1871-73), p. 178-193.

BRUNO CHRISTOFFEL (1829-1901), at that time at Zürich, became, after the Franco-Prussian war, professor at the new German university of Strassburg, where he taught with REYE. He also published some papers on related subjects, as on geodesic triangles on a surface and the determination of a surface by its mean curvature. RUDOLF LIPSCHITZ, (1832-1903) since 1864 professor at Bonn, who is also well remembered in analysis, continued his investigations on Riemannian manifolds during the next years and built up a theory of these manifolds; since RICCI's work these papers are somewhat antiquated. Both geometers established theorems on the equivalence of quadratic forms and derived the so-called Riemann-Christoffel tensor. To this purpose CHRISTOFFEL constructed his "CHRISTOFFEL symbols."

HERMANN HELMHOLTZ' (1821-1894) paper is *Ueber die Tatsachen, die der Geometrie zum Grunde liegen* (1868). (41) The name already shows the influence of RIEMANN's address of 1854, that was published a short time before HELMHOLTZ composed his paper. He emphasizes the empirical aspects of our space conception, in the way RIEMANN had indicated. HELMHOLTZ, a physiologist, had been led to the same idea by his own investigations. Experience, he says, gives us several measurable quantities, continuous and of more dimensions, such as space, the system of colors, and the field of vision. Each more-dimensional continuum is characterized by its own special properties, and our task is to analyze these properties. HELMHOLTZ now gives four postulates by which he can get RIEMANN's geometry with the quadratic line element; they are : 1) the existence of n dimensions and of continuity; 2) the existence of moving rigid bodies; 3) the free movability of a rigid body; 4) no dependence of the form of a rigid body on rotations (the monodromy). He then proves as a purely mathematical theorem that RIEMANN's geometry is the only case in which these conditions are satisfied.

This paper by HELMHOLTZ (later corrected by LIE) was of great importance for the final adaptation of non-Euclidean geo-

(41) *Gött. Nachrichten* 1868, Nr. 9; *Wiss. Abhandlungen* II, p. 618-639. It was in 1866 preceded by an address "Ueber die tatsächlichen Grundlagen der Geometrie," *Wiss. Abh.* II, p. 610-617. Later HELMHOLTZ defended his thesis against the philosopher LAND: "Ueber den Ursprung und Sinn der geometrischen Sätze." *Mind* 10 (1878), p. 212-224, *Wiss. Abh.*, II, p. 640-660.

metry. But, like RIEMANN's paper it was more than that. It was one of the outstanding contributions to an understanding of the nature of space, which was brought from the field of speculation into that of experimental physics. As such it is one of the outstanding contributions to a materialistic conception of space.

DARBOUX' work will be discussed in the next paragraph. To this period belong also investigations of ALFRED ENNEPER (1830-1885), from 1859 Dozent, after 1870 professor at Göttingen. He gave his name to a minimal surface and to a theorem on the torsion of the asymptotic lines. The first papers by H. A. SCHWARZ (1843-1921) on minimal surfaces appeared after 1865; his final solution of the isoperimetrical problem for the sphere dates from 1884.

The ideas of LAMÉ were taken up in a different way by French geometers who now began to study triply-orthogonal systems. LAMÉ, following DUPIN, had used only such systems as enabled him to integrate his differential equations of physics. The question of the construction of such systems was still unsolved. BONNET, in a paper of 1862, opened it anew; then follow A. RIBAUCOUR (1845-1923), E. COMBESCURE (1824-1889), and DARBOUX (thesis), (42) who showed respectively the relation of these systems to circle congruences, the way in which such systems can be obtained by a transformation of a surface into other surfaces with parallel tangent planes, and their dependence on a differential equation of the third order.

To the work done in the older line we mention papers written by LOUIS AOUST, a priest, " chanoîne de Montpellier " (1814-1885), who was professor of mathematics at Marseilles since 1854 and contributed largely, in papers and textbooks, especially between 1860 and 1880, to the theory of curves and surfaces; in these his " courbure inclinée " plays a large, too large, a role.

REINHOLD HOPPE (1816-1900), in 1859 Dozent, and after 1870 professor at Berlin, also published many papers, most of them antiquated now; several however are of importance as first ventures into the differential geometry of n dimensions. His work on intrinsic coordinates of curves deserves mention.

(42) See the discussion by E. SALKOWSKI. *Encykl. d. math. Wiss.* III, 3, p. 541 See for the work done in France J. BERTRAND. Rapport sur les progrès les plus récents de l'analyse mathématique. Paris, 1867.

Several textbooks appear at this time. We already have mentioned PAUL SERRET's book. In 1868 appeared PETERSON's monograph *Über Kurven und Flächen*, in which the Moscow mathematician published several interesting results on applicability of surfaces (see footnote 6c). In O. BÖKLEN's *Analytische Geometrie des Raumes* of 1861 is a discussion of infinitesimal geometry. WILHELM SCHELL's (1826-1904) *Allgemeine Theorie der Kurven doppelter Krümmung* of 1859, was written under the influence of JACOBI, whose lectures on differential geometry he had heard in the winter of 1849-50. Some of these books were more monographs on certain subjects than complete textbooks. A good complete textbook on differential geometry had not yet been written.

10. — *Differential geometry from* 1870 *to* 1900.

From 1870 to the world war there were no important economic disturbances in Western Europe. Life at the expanding and flourishing universities went on very regularly, and mathematicians could devote their full attention to professional problems. A result of this long period of quiet development was a great progress in differential geometry, mainly along paths blazed by the previous generations. At the same time geometry became more and more an abstract science, which did no longer easily reveal its origin in practical problems. Most geometers of this period adopted the attitude of JACOBI that science exists for the glory of the human mind and that its primary importance lies outside the field of the applications. To this attitude we owe the development of differential geometry along lines of abstract beauty often comparable to that of the most unpractical of all sciences, the theory of numbers. France, Germany and Italy remained the chief centers, where at the universities differential geometry was taught and professed as a regular part of the schedule. Other countries participated in the development, sometimes in a very important way, as Norway with S. LIE, Sweden with A. BÄCKLUND, and Russia with P. L. TSCHEBYCHEF. By the end of the century the United States begins to exert some influence. It is impossible to give a discussion of all the different subjects taken into consideration, of the theories developed, and of the papers published during this period. This would, moreover, repeat in an unsatisfactory way what volume 3[3]

of the *Encyklopädie der mathematischen Wissenschaften* in a series of monographs has done very effectively. We shall only point out some very general trends.

At the opening of the period we have the early work of DARBOUX, KLEIN and LIE, influenced by C. JORDAN and the so-called Erlangen Program of KLEIN (1872). (43) Here the consequence is drawn from the geometrical investigations of the nineteenth century, and the group concept is found to be the underlying idea. A geometry is defined as the theory of invariants belonging to a given continuous group of transformations. Also in differential geometry this idea of transformation comes more and more into prominence. It finds its nucleus in EULER, and later in the work of MONGE, CARNOT and AMPÈRE, it arises again in the work of GAUSS, LAMÉ and RIEMANN, and finally in that of BELTRAMI. The Erlangen Program became the program indeed of almost all further work on geometry in the nineteenth century, revealing itself not only in opening of new fields of investigations, but also in the introduction of formal invariant methods into the exposition of the material.

Nevertheless, the Erlangen Program did not exhaust the field of geometry as a whole, and so of differential geometry. There have always been tendencies leading outside of the vigorous scheme by which KLEIN so successfully tried to lead the development of geometry. RIEMANN's manifold conception was broader, and Riemannian geometry can only fit into a wider frame than that of the Erlangen Program.

The three great geometers whose work dominates the period after 1870 are GASTON DARBOUX, SOPHUS LIE and their younger contemporary, LUIGI BIANCHI. LIE's work consciously lies in the school of the Erlangen Program, for which he is partly responsible. BIANCHI and DARBOUX, though their work belongs to the type considered by KLEIN, did not find their field of work in problems especially suggested by it. The conscious cultivation of the differential geometry of different continuous groups belongs principally to the twentieth century.

G. DARBOUX's (1824-1917) career is analogous to that of many

(43) F. KLEIN, Vergleichende Betrachtungen über neuere geometrische Forschungen, Programm Erlangen 1872, *Math. Annalen*, 43 (1893), p. 63-100.

French geometers of the nineteenth century, his life being that of scholar and teacher at Paris. A remarkable fact is that for his education he was one of the first to prefer the Ecole Normale to the Ecole Polytechnique. In his later years, as secrétaire perpétuel of the Academy he exercised influence far beyond the limits of his special field of work. His work, even more than that of L. BIANCHI (1856-1928), professor of mathematics at Pisa and pupil of BETTI and DINI, follows the classical lines and shows the enormous variety of problems left unsolved by MONGE, LAMÉ and others. DARBOUX's early work (beginning 1866) deals with orthogonal sets of surfaces, each family of surfaces necessarily depending on a partial differential equation of the third order. Then he passes to a study of the deformation of surfaces, to the applicability of surfaces, and to surfaces of constant curvature. BIANCHI's productive career begins with the applicability of surfaces (1878) and in rapid succession follow papers on surfaces of constant curvature, orthogonal surfaces, Weingarten surfaces, minimal surfaces, and congruences. BIANCHI also turns to non-Euclidean geometry, taking up the ideas of BELTRAMI; the surfaces of curvature zero in such geometry draw his special attention. Following BÄCKLUND he invents transformations to pass from one set of surfaces of special character to another. In the work of DARBOUX and BIANCHI partial differential equations play an important role. As a new aspect of the theory of curves on surfaces, DARBOUX introduces the coordinates x, y, z, of a surface as solutions of an equation

$$\frac{\delta^2\theta}{\delta\alpha\delta\beta} + A\frac{\delta\theta}{\delta\alpha} + B\frac{\delta\theta}{\delta\beta} + C\theta = 0,$$ A, B, C functions of α,β,

α=const, β=const appear then as conjugate parameter lines. In DARBOUX's work another characteristic appears, the " trièdre mobile," by which kinematical considerations are introduced into the study of curves and surfaces. This field, opened by CODAZZI, was especially cultivated by A. MANNHEIM (1831-1906).

The principal fame of DARBOUX and BIANCHI lies in their beautiful textbooks, in which they combined their own results with those of their predecessors. BIANCHI's *Lezioni di geometria differenziale* (1893), a new edition of autographed lectures published in 1886, is a systematic treatise of the theory of curves and surfaces, with special attention paid to more-dimensional geometry.

It became known in wider circles through the German translation of M. LUKAT (1899). DARBOUX's *Leçons sur la théorie générale des surfaces* (four volumes, 1887, 1889, 1894, 1896) is much more than the title professes; it is an exposition of a vast number of theories related to the theory of surfaces, connected in a loose way, so as to give the author an opportunity to demonstrate the enormous breadth of his knowledge and the elegance of his methods.

For SOPHUS LIE (1842-1899) the transformation group is not an important means of investigation alone; it is rather the central part of geometry. His work is closely connected with that of KLEIN. He was a clergyman's son from a village in Norway, and he visited Paris in 1869-70 together with KLEIN, where he exchanged ideas with C. JORDAN and DARBOUX. JORDAN, at that time, had taken up GALOIS' result and published the *Traité des substitutions*. During their travel the fundamental importance of the group became clear to KLEIN and LIE. A first result was a joint paper on W-curves, plane curves invariant under a projective transformation of the plane. The Erlangen Program cast their ideas into permanent form. LIE then started out on his life work, the study and classification of continuous transformation groups, an inquiry which at the same time opened a new road to the study of partial differential equations. Regularly he reached results of importance to differential geometry. We already had occasion to compare his combination of geometrical intuition with analytical skill to that of MONGE. In his hands more-dimensional considerations grew into as natural a part of geometry as the classical methods. Of his special contributions we may mention the duality between line and sphere geometry leading to the duality between asymptotic curves and lines of curvature, the integration of the FRENET formulas if a relation between curvature, torsion and arc length is given, his work on minimal surfaces and contact transformations. He improved HELMHOLTZ' analysis of RIEMANN's problem of space. Of his older pupils we mention F. ENGEL and W. KILLING.

More-dimensional differential geometry found also investigators following the directives of RIEMANN, CHRISTOFFEL and BELTRAMI. Through a series of investigations undertaken by many authors a complete theory of curves and higher manifolds was developed. This study was considerably simplified by the invention of the

" calculo differenziale assoluto," a symbolism to express the invariants of Riemannian geometry. The inventor was GREGORIO RICCI-CURBASTRO (1853-1924), as BIANCHI pupil of BETTI and DINI at Pisa, since 1880 professor at Padua. RICCI published his discoveries in a series of papers beginning in 1886.

An entire new field is represented by the work of HENRI POINCARÉ (1854-1912). In a series of papers under the title : *Mémoire sur les courbes définies par une équation différentielle*, published in the *Journal de mathématiques* from 1881-1886, the great physicist, astronomer and analyst undertook the study of the properties of integral curves of ordinary differential equations, their singular points and their behavior in the large. (44) Again differential geometry was enlarged by influences from outside fields, as in the times of GAUSS and of LAMÉ. POINCARÉ was led to his theory by his investigation on problems similar to the three-body problem and related questions of dynamics, where geodesics appear as trajectories of moving particles. Here appear singularities like the " nœud," the " col," the " centre ", and the " foyer," and relations connecting them. Analysis situs becomes definitely connected with differential geometry. Differential geometry in the large, so far appearing only in isolated remarks of EULER, GAUSS, JACOBI, MINDING, turns into an important and difficult science of its own. The question of surfaces with closed lines of a certain kind arises.

The main tendencies of the years 1870-1900 can be summarized as follows :

a) consequent study of continuous transformation groups;

b) triply orthogonal systems;

c) surfaces of constant curvature;

d) applicability and deformation of surfaces;

e) renewed study of partial differential equations and their geometrical interpretation.

f) integral curves of ordinary differential equations;

g) more-dimensional geometry;

h) adaptation of invariant methods, as vector analysis or absolute differential calculus;

i) conception of comprehensive textbooks related to the general

(44) See the discussion by H. LIEBMANN, *Encykl. d. math. Wiss.*, III, 3, p. 503.

acceptance of differential geometry in the mathematical programs of the universities.

There are, of course, many aspects to this rich development that we have failed to emphasize or even to mention. The work of E. CESARO (1859-1906), professor at Naples, on natural geometry, resulting in a well-known textbook (1896) belongs to these. It may be characterized as one of the attempts to introduce invariant methods. Other mathematicians began to introduce vector methods into differential geometry, as G. PEANO (1887), or C. BURALI FORTI (1897); other attempts, like that of J. KNOBLAUCH (1888) may also be classified under this heading.

A regular output of textbooks showed how general differential geometry was taught. Almost all the " Cours d'analyses " and corresponding books in other languages devoted chapters to curves and surfaces. Since the seventies special books on the subject appeared regularly, from the books by the Abbé AOUST to those of L. RAFFY (1897), G. RICCI (1898) and W. DE TANNENBERG (1899). By the end of the century differential geometry had begun to break up into special branches, each of which had its specialists.

This leads us to the beginning of the twentieth century, when other tendencies begin to exert influence. We shall not deal with them in this present outline.

11. — *Sources.*

The material, necessary for a history of differential geometry, is scattered over many publications, and so far as we know has never been reviewed as an historical whole. Up to 1800 we have V. KOMMERELL's excellent paper on *Raumkurven und Flächen* in the fourth volume of M. CANTOR's *Vorlesungen über die Geschichte der Mathematik* (1908). After 1800 we have F. KLEIN's monumental book on *Entwicklung der Mathematik im 19. Jahrhundert* I (1926), but this deals with only a few aspects of differential geometry. These are the only larger expositions on the subject. Of importance also are A. HAAS, *Versuch einer Darstellung der Geschichte des Krümmungsmasses* (Diss. Tübingen, 1881), not always correct, and S. A. CHRISTENSEN, *Om den historiske udvikling af theorien for fladers og rumkurvers krumning* (Tidskrift f. Math., 1,

1883, p. 97-127). Then there are several monographs, as P. STÄCKEL, *Bemerkungen zur Geschichte der geodätischen Linien*," (Leipz. Berichte 45, 1893, p. 444-467), (45) and many biographies, of which the " Eloges " of ARAGO, BERTRAND and DARBOUX contain fascinating details, and are both valuable and charming.

12. — *Final remarks.*

From time to time we have been able to refer, in a few words, to the influence of external factors in the development of differential geometry. There is a general tendency here.

Nobody seems more free than the mathematician in the selection of his problems; yet even he in this respect has no complete liberty. He generally follows the directives of a certain school of thought, opened by a leader in the field, and is guided in his work by established traditions. The leader, however, is himself under certain influences in the selection of his material, and the direction of his labor is, as a rule, determined by external factors. In all cases his topics of research are in some way or another connected with the material conditions under which his generation lives, either directly by the necessity of the mathematical treatment of a technical problem, or indirectly by certain philosophical principles estimated as valuable in his time. Even when freedom of selection seems the very nature of his mathematical work, it is material circumstances which allow the required amount of abstraction from daily needs, as in the later part of the nineteenth century under the " science for science's sake " attitude prevalent at many universities, and reaction is bound to follow closely. We, therefore, find a subtle, but nevertheless definite, relationship between the general economic problems that humanity has had to solve and the investigations of the mathematician. It is, therefore, possible to indicate these connections even in the case of one of the most abstract fields of mathematics, the application of analysis to geometry. Though this field is too large to admit more than a very insufficient treatment in a few pages, we were nevertheless

(45) Also a master's thesis at the Mass. Inst. of Technology by J. L. LAWSON. The history of the theory of curves and surfaces as contained in papers presented before the French Academy of Science during the entire eighteenth century, 33 p., 1931.

able to give some indications of how the different schools of geometers have been influenced by the technical requirements of their age.

Differential geometry owes its results to such problems as are set by map projection, surveying, measurement of time (HUYGENS), fortification and other problems of warfare (MONGE); these are supplemented by problems taken from potential theory, elasticity, light and vision. Its very origin is due to the invention of the calculus, a result of a long series of involved technical problems in a period when geometry was still the main form of mathematics. The French revolution opened enormous sources of energy for its promotion, and influenced that renewed study of space and time connected with the names of KANT, GAUSS and LOBATCHEVSKY. The steady penetration of materialistic ideas into the study of separate branches of science in the course of the nineteenth century resulted in the realistic analysis of the space problem by RIEMANN and HELMHOLTZ, and removed the previously existing barriers to the study of n-dimensional geometry. For the organization and further cultivation of differential geometry the nationalistic revival of the sixties was of great importance, as is shown, for example, by the Risorgimento in Italy. And the unhampered development of science in the latter part of the last century, which resulted in a steady growth of our knowledge of curves and surfaces, also finds its source in the undisturbed material conditions of Europe during that time.

These are a few of the tendencies that have been responsible for the development of one of the most fascinating branches of modern mathematics. In the preceding exposition we had to make a selection, which was to a certain extent arbitrary, and which depended on our personal preferences. Nevertheless, we hope we have followed what most mathematicians will consider the main lines of development.

Massachusetts Institute of Technology
Cambridge, Mass.

February 1932.
D. J. STRUIK.

The history of mathematical time

II *

V. — *The Twentieth Century; Time in Modern Mathematical Physics*

Modern physics is essentially mathematical. It also makes considerable use of the conception of Time, which enters as an important factor in the new mechanics, replacing in part the Newtonian conceptions upon which our knowledge of the universe has been built. Whether our conceptions of Time and Space have received adequate examination in view of their changed significance in the mathematical scheme of things is a debatable point, the importance of which should not be overlooked, but which does not concern us at present.

In facing the problem of Time in modern mathematical physics the historian is faced with two difficulties; the necessity of reviewing the development of a science which has not yet reached finality, or even agreement, between its different parts, and the necessity for regarding all the modern implications concerning Time entirely apart from the physical views which accompany them in the complete system.

The attempt at a preparation of anything approaching a complete history of Mathematical Time as represented by modern physics would at present be premature for these reasons. The most we can therefore attempt is an account of the subject in its historical order of development. The nature of the subject requires that criticism should be either suspended or relegated to a second place.

Influence of Dynamics upon the Idea of Time. — The advances

* See *Isis*, 19, 121-153.

in the science of mechanics which took place during the nineteenth century may be considered as due to contemporary advancements in mathematical thought. In the early part of the twentieth century these advancements reached a high stage of development in the evolution of a completely new conception of physical phenomena. This new conception may be regarded as due to the attempt to fit physical phenomena into a highly developed system of mathematics, instead of endeavouring to apply existing mathematics to the explanation of observed phenomena.

The classical dynamics of GALILEO and NEWTON is based on the conception of Time as being something that flows uniformly, in accordance with NEWTON's definition. This definition, however, is not conceptually valid, on account of the fact that the definition of flow involves that of Time as a necessary component, and hence imparts circularity to the definition. There is, however, no objection to it as a means of describing the nature of the accepted temporal continuum, as it involves no analytical error in the solution of such problems as are amenable to analysis by the equations of the classical dynamics.

An interesting discussion of Space and Time which illustrates this viewpoint is given by BARTHOLOMEW PRICE, Sedleian professor of Natural Philosophy and Fellow of Pembroke College, Oxford, in Vol. III of his *Treatise on Infinitesimal Calculus*. (19) In developing his treatment of the motion of material particles, Price gives a detailed and valuable discussion of " Motion, its Affections, its Laws, and its Equations." He remarks that such symbols as t, dt, s, ds, are *numbers* and not the concrete things themselves. " Thus t expresses the t times an unit of time is taken, dv the dv times an unit of velocity is taken; and the numbers of course can be multiplied together, and the resultant of the operation is number of that kind which the symbols express before the operation." With regard to matter, when the property of mobility is considered " there are two other affections, which, by reason of their abstract nature, need not be defined; viz : *time* and *space;* it is sufficient for us to be able to form a notion of them, and to enuntiate of them such properties as are required

(19) The entire work consists of four volumes, of which the first two are now out of print. We are using the 1856 edition, Ch. VII of Vol. III.

for the science of motion." Following upon this statement, PRICE gives a further confirmation of the views instituted by BARROW and NEWTON : " Space and time, like matter, are continuous and divisible; and these affections are without limit. Space may be very large, nay infinite; we cannot fix the boundaries of that space in which the heavenly bodies are; and it may be very small, such as that occupied by a chemical atom or a material particle. Time also admits of degrees as to quantity; it may be an instant; such an infinitesimal, that the aggregate of an infinity will make only finite time; or it may reach through the present moment from ages bygone to ages to come." The mathematical notion of Time is as clearly fixed with PRICE as with BARROW : " The notion of equal times and also the measure of equal times arises out of the idea of time, and an idea of time is not adequate unless it has these notions... they exist before it and they enable us to apprehend and to measure... equal times therefore must be deduced from the notion of time." The entire structure of the classical mechanics with its notions of velocity and acceleration, rests upon the fundamentals which are here outlined so clearly by PRICE, whose views may be regarded as representative of those of the Newtonian school.

Modifications of the Classical Mechanics. — At the close of the nineteenth and early part of the twentieth century, mathematicians and physicists began to consider the effects produced by motion of matter through the ether of space. Laboratory experiments had shown that there were numerous phenomena which could not be explained by means of the classical ideas of mechanics. The famous MICHELSON and MORLEY experiment of 1887, (20) which was conducted to observe the difference in the times taken by a ray of light to travel a given distance when the path of the ray was parallel or perpendicular to the motion of the earth in its orbit, showed a null effect : No difference was observed. This fact led to a great amount of mathematical work, and much scientific speculation, in which a prominent part was taken by H. A. LORENTZ, who examined the electrical and optical phenomena of motion in various papers published towards the end of the nineteenth century. (21) A useful account of the various

(20) *Philosophical Magazine*, Dec. 1887.

(21) See, e.g. H. A. LORENTZ, "De l'influence du mouvement de la Terre sur

developments in the theory up to about this time were given in connexion with a discussion at a meeting of the German Scientific Association by W. WIEN. (22)

One of the implications of LORENTZ's analysis is that different moving bodies require a different system of time measurement for the expression of their phenomena. This necessity is additional to the contraction hypothesis which was put forward by G. F. FITZGERALD in 1893 and independently by LORENTZ in 1895 to account for the null effect of the MICHELSON-MORLEY experiment. The adoption of this arrangement involves the idea that every moving system has its own " local " time : an idea which lies at the foundations of the celebrated Theory of Relativity. From the mathematical work of LORENTZ in this connexion several important consequences may be deduced, (23) merely from the general geometry of Time and Space. These consequences are such as to cast grave doubts upon the accuracy of the ordinary conceptions of Time, with its attendant notions of *simultaneity* of events, and *interval* between events.

If we postulate that electromagnetic phenomena occurring in moving systems are exactly the same as those in stationary systems, then the theory of LORENTZ implies that we must not only adopt the contraction hypothesis, but also the further hypothesis that the measures of Time in different moving systems are different. The results of this theory are at variance with those of the classical mechanics : They lead to the group of theorems which are included in the theory of Relativity.

Valuable observations on the theory of LORENTZ were made by HENRI POINCARÉ (1854-1912) in a paper *Sur la dynamique de l'électron*, (24) where he enquires into the theory of measurement, and clearly shows the difference between the old systems of measurement and the new : We measure by superposing one on the other objects which may be regarded as rigid and invariable,

---

les phénomènes lumineux, " *Archives Néerlandaises*, 21, 1887, pp. 103-176 ; " La Théorie Électromagnétique de MAXWELL, et son application aux corps mouvants," *ibid.*, 25, 1892 ; " Versuch einer Theorie der elektrischen und optischen Erscheinungen in bewegten Körpern," Leiden, 1895.

(22) *Wiedemann's Annalen der Physik*, 65, July 1898.

(23) See e.g. Sir ARTHUR STANLEY EDDINGTON " The Mathematical Theory of Relativity," Cambridge, 1923, § 10.

(24) *Rendiconti del circolo matematico di Palermo*, 21, 1906, p. 3.

but if we accept the LORENTZ contraction hypothesis this is no longer true. In this theory the criterion of equality of two lengths is that they shall both be traversed by light in equal times. Here POINCARÉ makes an interesting simile : " To give up this definition would perhaps be sufficient to overthrow the theory of LORENTZ, as completely as the Ptolemaic system was overthrown by COPERNICUS. If this should happen, it will not prove that the effort made by LORENTZ has been useless. PTOLEMY, whatever one thinks of him, was useful to COPERNICUS."

The difficulties attending the electrodynamics of moving bodies received the attention of ALBERT EINSTEIN in a paper *Zur Electrodynamik bewegter Körpern*, (25) which was published in the same year as POINCARÉ's paper was read. EINSTEIN considers here that the difficulties encountered by the theory of electrodynamics in moving systems are traceable to a lack of consideration of the fact that the theory is based upon the kinematics of the rigid body which in turn is based upon relationships between rigid bodies, clocks and electromagnetic processes.

The importance of the true significance of Time in kinetic phenomena is stressed by EINSTEIN in the early stages of his paper, and forms an essential part of the principle of Relativity which is therein developed. Much difficulty and cause for speculation centres round the idea of simultaneity : There is, for example, the problem of two systems which move in a straight line relatively to each other, and each carries an observer who considers his particular system to be at rest. Two events which appear simultaneous to one observer will appear nonsimultaneous to the other. This is a condition of which it is very difficult to grasp the significance if we retain the Newtonian conception of Time as that which " without relation to anything external flows equally." While there is no difficulty in realizing the meaning of simultaneity between two events when they occur at the same place, it is difficult to form a clear idea of simultaneity between two events occurring in different places. As soon as we attempt to define simultaneity in the latter case we find it necessary to introduce the notion of light rays and chronometers, which add further difficulty on account of the necessity of being able to forecast exactly their

(25) *Annalen der Physik*, 17, 1905, p. 891 .

behaviour when used as time-measurers under these conditions.

The basis of EINSTEIN's first examination of the problem is the postulate that the velocity of light is constant, and entirely independent of the velocity of the moving system or its direction in space. If this postulate be applied to the problem of two systems in relative motion, one using space coordinates x, y, z and a time co-ordinate t and the other using space co-ordinates x_1, y_1, z_1 and a time co-ordinate t_1, then the space traversed by light in a given time is constant for either system and we have

$$\sqrt{x^2 + y^2 + z^2}/t = C \text{ for one system, and}$$

$$\sqrt{x^2_1 + y^2_1 + z^2_1}/t_1 = C \text{ for the other system,}$$

where C is the velocity of light which, by hypothesis = a constant. From the foregoing equations we obtain directly

$$x^2 + y^2 + z^2/t^2 = C^2,$$
$$x^2_1 + y^2_1 + z^2_1/t^2_1 = C^2, \text{ so that}$$
$$x^2 + y^2 + z^2 = C^2t^2,$$
$$x_1^2 + y_1^2 + z_1^2 = C^2t_1^2, \text{ and therefore}$$
$$x^2 + y^2 + z^2 - C^2t^2 = O,$$
$$x^2_1 + y_1^2 + z_1^2 - C^2t_1^2 = O.$$

It will be seen that we now have in place of the simple time co-ordinate t a less simple, in fact a somewhat puzzling, co-ordinate iCt, where $i = \sqrt{-1}$. It is interesting here to revert to ARTHUR CAYLEY's views on HAMILTON's memoir of 1833... " And still less can I appreciate the manner in which the author connects with the notion of time his algebraic couple, or imaginary magnitude, $a + b\sqrt{-1}$." There are evidently more things in mathematical physics than were thought of by ARTHUR CAYLEY, or indeed, than could very well have been thought of by a pure mathematician holding the Newtonian views of Time and Space.

The transformations giving rise to " imaginary " Time were made in a paper read in 1908 by the great mathematician, HERMANN MINKOWSKI (1864-1909) of Bonn and Königsberg. As a pure mathematician MINKOWSKI is famous for his work on whole numbers and in many other departments of mathematics. The paper to which we now refer was given in the form of an address *Der Raum und Zeit* before the 80th Assembly of German Natural Scientists and Physicians at Cologne on the 21st of September, 1908.

The correlation of Time and Space in the new physics of LORENTZ and EINSTEIN is stressed by MINKOWSKI in the prefatory remarks to this famous address, where he says that " Henceforth Space by itself and Time by itself are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality." For MINKOWSKI, the LORENTZ equations do not represent an artificial world or paradox, but indicate the connection between Space and Time which had hitherto received but scant attention. We cannot think in terms of Time and Space separately. " Nobody has ever observed a place except at a time, or a time except at a place." Thus MINKOWSKI introduces a revised metric, where a " world point " is represented by x, y, z, t, and the " world " is composed of all thinkable values of these variables.

Any point of space defined by x, y, z, at a definite point of Time t corresponds to the world point x, y, z, t. If variations of t are considered, then there is a corresponding variation in the three variables x, y, z, and the line which is accordingly described by the point is called its " world line." It follows that if the world lines of all point-events comprising physical phenomena could be specified we should obtain a complete history of the entire universe, both with regard to past and future.

The theory of MINKOWSKI has received the attention of FELIX KLEIN (1849-1925) of Göttingen, who pointed out that " what the physicists call the " theory of relativity " is the theory of invariants of the four-dimensional space-time region x, y, z, t (MINKOWSKI's world) in relation to a definite group of collineations, namely, the ' Lorentz Group.' " (26)

By putting $\tau = iCt$ in the equations developed above, MINKOWSKI expressed his " world line " in the form

$$S^2 = x^2 + y^2 + z^2 + \tau^2$$

which is exactly analogous to the invariant of Euclidean geometry, and in which Time as we have hitherto considered it has indeed " faded away into a mere shadow " of its former self. We observe here that modern physics, as represented by the theory of relativity, is not any further concerned with the attributes of what we have hitherto been pleased to call Time. For the modern mathematical

(26) *Jahresbuch d.d. Math. Verein.*, 19, 1910, p. 287.

physicist Time may be regarded as an attribute of Space, and has no independent existence. It is incorporated in a symbol which expresses its relation to Space through the medium of a hypothetical constant of the universe, the velocity of light. This disappearance of pure mathematical Time prevents any further pursuit of it, in so far as the theory of Relativity is concerned. We therefore turn our attention to further developments, of which many may be regarded as the offspring of the theory of Relativity.

A fanciful example of the consequences of this theory has been given by PIERRE LANGEVIN. (27) A ball, containing a clock and an observer, is propelled from the earth with a velocity approaching the velocity of light. In this moving system the time, and all natural processes, go more slowly, so that when the ball returns to earth after two years measured by its chronometer, the traveller will see an earth whose face has been changed by the lapse of two or three hundred years as measured by earthly time systems. The paper of LANGEVIN containing this idea, which has been developed by EDDINGTON and others, (28) is mentioned in the preface of the late H. WILDON CARR's " General Principle of Relativity," (29) in connexion with his purely philosophical interest in the principles of relativity. " I first became acquainted with it at the International Congress of Philosophy at Bologna in 1911, when M. PIERRE LANGEVIN, Professor of the Collège de France, revealed its philosophical importance in a remarkable paper..."

Also in the year 1911 appeared a short tract by ALFRED A. ROBB entitled *Optical Geometry of Motion, a New View of the Theory of Relativity*. This tract is of importance in containing the fundamentals from which the same author subsequently developed a more comprehensive physico-mathematical theory of Time than had hitherto been attempted. The identification of instants at different places is carefully avoided by ROBB, both in his preliminary tract and in the comprehensive treatise which followed. The

(27) In a paper on " L'Évolution de l'espace et du temps " *Scientia*, 10, 1911, p. 50.

(28) See e.g. *Space, Time and Gravitation* by A. S. EDDINGTON, Cambridge, 1923.

(29) London, 1920, p. vi of Preface.

geometrical conception of ' Conical Order ' forming the basis of this later work treats of instants at different places as being definitely distinct.

A brief account of the idea of ' Conical Order ' was published by ROBB in 1913. (30) A larger work, tracing more fully the implications of this idea, was published in the following year. (31) A ' conical order ' is used for representing the Space-Time continuum. From 21 postulates, derived entirely from the conceptions of *before* and *after* is developed a system of geometry in which any point-event, corresponding to MINKOWSKI's ' Weltpunkt,' can be represented by the coordinates x, y, z, t. This geometry leads to the inclusion of Time as a fourth coordinate inherent to the system. There is no independent theory of pure Time.

Certain of the principles upon which ROBB's theory is built are fundamentally at variance with the space-time hypotheses of relativistic mechanics. It is pointed out by ROBB that if two physicists, A and B, agree to discuss a physical experiment, then it follows, in virtue of their agreement, that a common world of some kind exists as the seat of the experiment. If it be urged that we have merely a correspondence between the physical worlds of A and B, then the exact nature of this correspondence remains to be explained. The fact that there is actually such a correspondence indicates that the phenomena are common to the minds of A and B, whence it follows that the phenomena are normal to some common reference continuum.

It is interesting here to compare this view with that of the *sensorium Dei*, characteristic of NEWTON's philosophy, which gave rise to considerable misunderstanding among contemporary philosophers. NEWTON's real contention is clearly shown in the following passage from the *Principia, Scholium Generale:* " Partes dantur successivae in duratione, coexistentes in spatio, neutrae in persona hominis seu principio ejus cogitante; et multo minus in substantia cogitante Dei."

The bases of ROBB's theory of ' conical order ' are stated as follows : (31) " An element of time is called an instant and is to be regarded as a fundamental concept.

---

(30) *A Theory of Time and Space*, Cambridge (HEFFER and Sons) 1913.
(31) *A Theory of Time and Space*, Cambridge (University Press) 1914.

Of any two elements of time of which I am *directly* conscious, one is *after* the other.

The relation of two instants, one of which is *after* the other, is an asymmetrical relation, and the converse asymmetrical relation is denoted by the term *before*, so that if an instant A be *after* an instant B, the instant B is *before* the instant A."

The development of this concept then takes place in a manner which reminds us of the work of HAMILTON to which we have already referred, except in that the theory now under discussion is essentially geometrical, whereas that of HAMILTON was purely algebraical. ROBB builds his theory on the principle that " the set of instants of which I am directly conscious form a series in linear order. Thus they satisfy the following conditions :

(1) If an instant A be *after* an instant B, the instant B is not *after* the instant A, and is said to be *before* it.

(2) If A be any instant, I can conceive of an instant which is *after* A and also of one which is *before* A.

(3) If an instant A be *after* an instant B, I can conceive of an instant which is both *after* B and *before* A.

(4) If an instant B be *after* an instant A and an instant C be *after* the instant B, the instant C is *after* the instant A.

(5) If an instant A be neither *before* nor *after* an instant B, the instants A and B are identical."

The conception of ' conical order ' leads to the important idea that the only events which are really simultaneous are those which occur at the same place. Although the time-continuum of an individual is composed of instants which are in linear order, the aggregate of all instants, i.e. the universal time-continuum, forms a set in conical order.

The geometrical system which is built up from this idea is of a four dimensional character, in which any element of the system is determined by four coordinates. Thus once again we witness the disappearance of Time as a separate conception. On this occasion it becomes allied with space in forming part of a geometrical representation of the space-time continuum in which physical phenomena have their existence.

A small volume, introductory to the large work of 1914, was published by ROBB in 1921. In the " Preliminary Considerations" is a very valuable statement of the fundamentals of the conception

of Time. (32) It is the belief of ROBB that a neglect of these fundamentals is responsible for some inaccuracies of logic in the foundations of the principle of relativity.

An important and fundamental proposition arises from ROBB's theory as follows : Suppose we imagine a flash of light to be sent out at an instant A from a particle P to a distant particle Q, where it arrives at an instant B; and suppose it to be reflected back to P, where it arrives at an instant C : How are we to identify the instant B with any instant at P between A and C? This type of problem, which combines the notions of Space and Time, is representative of the difficulties which are encountered in applying the idea of Time to physical phenomena, in the manner which is attempted by the theory of Relativity.

The relativity principle was extended by EINSTEIN in two papers, *Die Grundlage der allgemeinen Relativitätstheorie* (33) and *Kosmologische Betrachtungen zur allgemeinen Relativitätstheorie* (34) which may be regarded as placing the theory on the plane from which subsequent developments have progressed. (35)

The underlying conceptions of simultaneity to which we have referred did not, and still do not, find universal acceptance among theorists. Thus LORENTZ, in his Haarlem Lectures of 1917 remarks : " I personally still find a certain satisfaction in older views that the ether still preserves a certain substantiality, that Space and Time are sharply distinguished, and that one can speak of simultaneity without a closer specialisation." (36)

A system of time, related to the concepts of the relativity principle, has been developed by A. N. WHITEHEAD. The system introduces the conception of duration as a temporal stratum of nature, limited in its temporal dimension and of unlimited spatial dimensions. (37) This system thus represents a finite

---

(32) A. A. ROBB, *The Absolute Relations of Time and Space*, Cambridge, 1921, pp. 1-16.

(33) *Annalen der Physik*, 49, 1916, p. 769.

(34) *Berlin Sitzungsberichte*, 1917, p. 142.

(35) A useful bibliography is given by EDDINGTON, *The Mathematical Theory of Relativity*, Cambridge, 1923, pp. 241-2.

(36) See the *Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht*, 1917.

(37) A. N. WHITEHEAD, *An Enquiry concerning the Principles of Natural Knowledge*, Cambridge 1919, Ch. 9, pp. 110-120.

Time and infinite Space, and is graphically expressible, on the assumption of unidimentional Space, by a straight line, representing Time, normal to a system of straight lines, each of which represents Space at the point of Time corresponding to the point of its intersection with the Time-axis. The assumption of manifold Time-systems, in accordance with relativistic mechanics, imparts to this spatio-temporal system a complex structure involving the intersection and partial congruence of durations in different Time-systems, and also introduces logical difficulties in connexion with the properties of durations.

The system reminds us of KANT's suggestion, to which we have already referred, in his Inaugural Dissertation of 1770... " For if we represent Time by a straight line extended to infinity, and simultaneous things at any point of Time by lines successively erected perpendicular to it, the plane thus generated will represent the phenominal world both as to its substance and as to its accidents." (§ 14, (5) *nota.*)

The problem of Time in its relation to the relativity principle was discussed by many writers, and from many different standpoints. We quote below some of the more important works and papers on this subject from the immense amount of material forming the literature of the theory of Relativity. (38)

An Article on " The Relativity of Time " which gives views of interest from the present standpoint was published by EDDINGTON (39) in 1921. It is here pointed out that whereas the astronomer is primarily concerned with the measurement of Time, the philosopher discusses its significance, and the two rarely think of comparing their ideas. It is consequently understandable that confusion should have arisen. The discovery of the finite velocity

(38) E. COHN, " Physikalisches über Raum und Zeit." *Abh. und Vortr. aus dem Geb. der Math., Naturwiss. und Tech.*, Heft 2, Leipzig, 1916.

H. WEYL, " Raum, Zeit, Materie," Berlin, 1918.

A. S. EDDINGTON, " Space, Time and Gravitation," Cambridge, 1920.

MORITZ SCHLICK, " Raum und Zeit in der gegenwärtigen Physik," Berlin 1920 (3rd ed.). English trans. by H. L. BROSE, Oxford, 1920.

L. SCHLESINGER, " Raum, Zeit und Relativitätstheorie," *Abh. und Vortr. aus dem Geb. der Math., Naturwiss. und Tech.* Heft. 5, Leipzig, 1920.

HANS WITTE, " Raum und Zeit im Lichte der neueren Physik " Heft 17 (2nd ed.), Braunschweig, 1920.

I. SCHNEIDER, " Das Raum-Zeit-Problem bei Kant und Einstein," Berlin, 1921.

(39) See " *Nature* " Special Number " Relativity," 106, 1921, p. 802.

of light by the astronomer RÖMER led to the abandonment of the idea which follows from our sense of vision that we are actually present at events far distant from us. The progress of astronomy has shown that this conception is wrong. As EDDINGTON points out : " Since RÖMER, time has become a mathematical construction devised to give the least disturbance to the old illusion that the instants in our consciousness are world-wide."

This point is pursued by EDDINGTON in his classical treatise on Relativity (35) where he says that " those who still insist on the existence of a unique " true time " generally rely on the possibility that the resources of experiment are not yet exhausted and that some day a discriminating test may be found. But the off-chance that a future generation may discover a significance in our utterances is scarcely an excuse for making meaningless noises."

Articles 1 to 18 of EDDINGTON's treatise may be regarded as probably the most consistent account which has yet been published concerning Time and space-time from the viewpoint of modern and mathematical physics.

In the year 1924 appeared an English translation of an important work by A. V. VASILIEV under the title " Space, Time, Motion." This work adopts the historical order of presentation, and is intended as an introduction to the theory of Relativity. It contains many valuable observations on the conception of Time from the mathematical viewpoint, and also many interesting historical notes. The conclusion of the work is significant : Referring to the problem of reality, which must be the ultimate aim of mathematical physics, the author touches on " that other riddle, why and wherefore an insignificant speck of cosmic dust gave birth to human thought. In the history of human thought the page devoted to the problems of space and motion is one of the most important and the most fruitful. For the solution of this problem let us take for our motto ' Laboremus '."

It appears from our researches into the historical side of the problem of Time that we must indeed exert great effort if we are to correlate the various theories which have been propounded up to the present, and therefrom to form a consistent view of Time which shall be true alike for physicist, astronomer, and philosopher. The problem is a great one, and recent develop-

ments show it to be of great scientific importance as well as of absorbing interest from the side of pure knowledge.

An example of modern development is afforded by an article on " Time Measurement in an Isotropic Space Frame " by J. L. SYNGE. (39*a*) Here temporal simultaneity is deduced from the postulate of the complete isotropy of space. From the fundamental theorem (which is also given by ROBB (39*b*)) that " If an event A is simultaneous with an event B, and B is simultaneous with a third event C, then A is simultaneous with C," is proved the essential theorem of the Transitivity of Simultaneity.

Numerous other papers which have a bearing on the subject of Time in one or other of its aspects may here be referred to. (39*c*)

VI. — *Recent Developments in the Theory of Time; Conclusion*

During recent years considerable progress has been made in methods for the measurement of extremely small physical entities. The dimensions of the atom have been established to what is probably a fair degree of accuracy, and the atom itself has been resolved theoretically into a nucleus accompanied by a system of orbital electrons, in some respects analogous to the solar system in which we have our being.

Transference of energy, such as radiation, is regarded by modern physics as taking place discontinuously, in little jumps. This view of energy transfer is due to MAX PLANCK, (40) and forms a natural corollary to the conception of the atomic universe.

For modern physics, therefore, neither matter nor energy are continuous : Both have a discontinuous structure which, to our

---

(39*a*) *Proc. Royal Irish Academy*, (A) Vol. 37, 1927, p. 110.

(39*b*) *Vide supra.* (31) (32).

(39*c*) COLLINGWOOD " Some Perplexities about Time, *Aristotelian Soc. Proc.* 1926.

WHITEHEAD " Time," *Proc. 6th Int. Cong. Phil.* (Harv.) 1926.

GOLDSBOROUGH " Recent Discussions on Time," 1927.

G. N. LEWIS " Symmetry of Time in Physics," *Science*, 71, 1930, p. 569.

J. W. TOMB. " An Essay on Metaphysical Time," London, 1930. (Reprinted from the *Transactions of the Indian Philosophical Congress*, Bombay, 1927); See " *Nature*," vol. 127, 1931, p. 104.

(40) *Annalen der Physik*, 4, 1901, p. 553.

imperfect senses, appears continuous. From this view of the universe arises the questions : " Are Time and Space continuous, or made up of atomic constituents ? "

At this point we may revert to BARROW's conception of Time as the " trace of a Moment continually flowing," as being composed of an infinite number of instants, just as in mathematics a line may be regarded as made up of an infinite number of mathematical points.

Recent researches in mathematical physics seem to indicate that there is at least a reasonable ground for supposing Time to be composed of discrete instants.

This idea was formulated by ROBERT LÉVY, who obtained for the value of the Time-atom, which he calls ' Chronon,' (41) the number $\Theta = 4.5 \times 10^{-24}$ seconds.

The work of LEVY was extended by G. I. POKROWSKI (42) in a paper " Zur Frage nach einer oberen Grenze für die Energiedichte," where it is shown that the average value of the shortest period of an electrical oscillation is

$$\Theta = 4.3 \times 10^{-24} \text{ seconds.}$$

which is in good agreement with the theoretical value. A further paper by POKROWSKI " Zur Frage nach der Struktur der Zeit," (42*a*) gives a proof of the accuracy of the value found for ' Chronon ' by comparing it with the shortest period of oscillation observable in the spectrum of short rays. In the concluding words of this paper : " Allerdings ist aus dem Gesagten klar, welche Wichtigkeit genaue Messungen der kürzesten Wellenlängen für die Entscheidung einer der interessantesten Fragen der Physik haben könnten."

Additions to the theory were made by GOTTFRIED BECK (43) WILHELM ANDERSON, (44) and SEITARÔ SÙZUKI. (45) These developments led to the establishment by E. REGENER (46) of a range of regularities bearing upon astrophysics and the nature of cosmic rays. An article " Zur Diracschen Theorie von Protonen und

(41) *C. R.*, 183, 1926, p. 1026.
(42) *Zeitschrift für Physik*, 51, 1928, p. 730.
(42*a*) *Zeits. f. Phys.*, 51, 1928, p. 737.
(43) *Zeits. f. Phys.*, 53, 1929, p. 675.
(44) *Zeits. f. Phys.*, 55, 1929, p. 386.
(45) *Physikalische Zeits.*, 31, 1930, p. 619.
(46) *Die Naturwissenschaften*, 17, 1929, p. 183.

Elektronen " by POKROWSKI (47) shows that these regularities are in good agreement with the theory of protons and electrons put forward by P. A. M. DIRAC (48), of Cambridge.

The element of Time in this theory is equal to h/MC^2, where M is the mass of the proton. A similar theory, in which the value of the Time element is h/mC^2 where m is the mass of the electron, has been suggested by F. O. WOLLASTON and K. W. MILLER. (49)

If Time possesses an atomic structure, we are left to make the generalisation that existence itself is atomic, or discrete. This theory would lead to an important revision of physics, and in the method of approaching physical phenomena by analysis. These speculations do not, however, form a part of our present enquiry.

From our considerations of the trend of modern physics it appears that there is considerable scope for further investigations of the framework of Time and Space in which we have our existence.

CONCLUSION

Pure and Applied Mathematical Time. — We are now arriving towards the end of our examination of the history of Mathematical Time. We have seen that the results comprising the present theoretical structure have been often widely diverse, both in character and method of treatment.

The mathematical theory of Time may be divided into two departments; pure and applied, corresponding to the two departments of mathematics. The purely mathematical theory of Time is represented by such researches as those of BARROW, HAMILTON and BUÉE, where the development is based on purely mathematical principles, and where there is no attempt to apply the theory to physical phenomena. The applied mathematical theory of Time, which is represented by the work of MINKOWSKI, LORENTZ and EINSTEIN, rests upon principles which are gathered by observation from physical phenomena. The applied theory therefore reflects the phenomena of the physical world.

(47) *Zeits. f. Phys.*, 66, 1930, p. 129.
(48) *Proc. Royal Society London* (A), 126, 1930, p. 360.
(49) *Nature*, 127, 1931, o. 163.

Any comparison between the pure and applied mathematical theories of Time must consequently involve the fundamental question as to how far the phenomena of our experience can be represented mathematically. In such abstruse subjects as that of the problem of Time the corellation of pure and applied theory presents grave difficulties. The pure theory must be regarded as the domain of the mathematician, and the applied theory as the domain of the physicist. It appears from our review of the respective theories that there remains much work yet to be done by both parties before any detailed reconciliation of their views is possible.

Applications of the Theory of Time. — (1) *Mechanics.* The conception of Time, along with those of Space, Matter and Force may be regarded as fundamental to the science of Mechanics. In the majority of works dealing with mechanics, however, very little attention is given to analysis of these fundamental conceptions. A notable exception to these works is represented by the *Mechanics* of ERNST MACH (1838-1916) professor of the History and Theory of the Inductive Sciences in the University of Vienna. MACH's *Mechanics* has formed the basis of many investigations in the underlying principles of mechanics, of which subjects it contains detailed discussions. A similar treatment is given by the Rev. JAMES CHALLIS, Plumian Professor of Astronomy at Cambridge, in his *Principles of Mathematics and Physics*, (50) where he says that " A system of Pure Calculation may be established by reasoning on abstract quantity antecedently to the consideration of properties of space, time, matter and force. The results of such reasoning may subsequently be applied to each of these kinds of quantity." This statement should, of course, be qualified by remarking upon the necessity of making quite certain that the transference from pure to applied theory is warranted and logical.

In stating the principles of Plane Astronomy, CHALLIS remarks that this science is founded on observations, and is therefore essentially practical : " There is also another distinctive feature which separates it from pure geometry, namely, the introduction of the element of *time*." The consideration of Time in astronomy

(50) Cambridge, 1869. The full title is *Notes on the Principles of Pure and Applied Calculation; and Applications of Mathematical Principles to Theories of the Physical Forces.*"

could be dispensed with only if all the heavenly bodies maintained at all times the same relative positions. The changes of position which actually occur introduce the necessity of an independent time-system. This requires particular attention, " because astronomical *measurements* of time and determinations of *epochs* are equally necessary in all other calculations which involve the consideration of changes which depend on the lapse of time. The science of *Time* is essentially a part of Plane Astronomy."

These views are also expressed by the great astronomer, Sir John F. W. HERSCHEL (1792-1871), Fellow of St. John's College, Cambridge, in his *Outlines of Astronomy*, (51) where he says that " Time is an essential element in astronomical observation, in a twofold point of view : 1st, As the representative of angular motion... 2ndly, As the fundamental element (or natural *independent variable*, to use the language of geometers) in all dynamical theories." There follows a discussion of astronomical Time and chronometers, which leads to the subject of determining longitude, and similar applications. Chapter XVIII " Of the Account of Time " may be regarded as expressing the viewpoint of plane astronomy : " Time, like distance, may be measured by comparison with standards of any length, and all that is requisite for ascertaining correctly the length of any interval, is to be able to apply the standard to the interval throughout its whole extent, without overlapping on the one hand, or leaving unmeasured vacancies on the other..." This view, of course, is that of the old mechanics. The conception of Time which underlies it is characteristic of the Newtonian philosophy.

We have seen that the theory of Relativity introduced new conceptions into the older views of Time and Space, and brought physicists to consider these subjects anew. It has been necessary for the physicist to introduce two fundamental postulates : That every point of Time, that is, every instant, is like every other instant, and that every point of Space is like every other point. This means that Time and Space can be regarded as homogeneous, (52) and the theory becomes analogous to the classical views

(51) London, 1878. There are some ten editions of this work. See also W. M. SMART, *Spherical Astronomy*. Cambridge 1931, Ch. VI., " Time."

(52) See e.g. ROBERT D. CARMICHAEL, *The Theory of Relativity* (Mathematical Monograph No. 12), New York, 1920 (2nd edit.) Ch. II.

in this respect. In the relativity theory, however, the three dimensions of Space and one dimension of Time are expressed as four variables which are definitely related to each other. This fact has given rise to the idea of Time as a " fourth dimension "; an unfortunate term which has led to the erroneous idea that Time by itself may be regarded as a physical dimension of the Universe.

The advancement of mathematical physics which followed as a result of the attacks made during the nineteenth century upon the fundamental conceptions of mechanics brought with it a more complex and involved method of examining physical phenomena. The physicist no longer concerned himself with the *meaning* or interpretation of his mathematical formulae and symbols. Thus he avoided many difficult questions, including that of the nature of Time. In the words of EDDINGTON : (53) " Nowadays time might be taken as typical of the kind of stuff of which we imagine the physical world to be built. Physics has no direct concern with that feeling of " becoming " in our consciousness which we regard as inherently belonging to the nature of time, and it treats time merely as a symbol; but equally matter and all else that is in the physical world have been reduced to a shadowy symbolism."

(2) *Electrical Theory*. The various departments of electrical theory make considerable use of the conception of Time. In those parts of the theory dealing with varying or oscillatory phenomena, such as are encountered in the theory of alternating currents and in electrostatics respectively, Time enters prominently into the calculations. (54) The time-differential is often encountered and is employed in the same manner as in theoretical mechanics and physics. The equations containing these terms are differentiated, giving a second differential of Time, i.e. the rate of change of a rate of change which so sorely puzzled Bishop BERKELEY in connexion with the philosophical aspect of the theory of fluxions.

It can be shown (55) that the electrostatic charge at every point in a homogeneous dielectric falls off exponentially with

(53) In the Swarthmore Lecture delivered before the Society of Friends, London, in 1929; *Science and the Unseen World*, London, 1929, p. 22.

(54) See e.g. Sir J. H. JEANS, *The Mathematical Theory of Electricity and Magnetism*, Cambridge, 1927, (5th ed.), Arts. 514, 516, 522, 537, 561, 583 and 583*a*.

(55) JEANS, *loc. cit.*, Art. 396.

the time, the modulus of decay corresponding to the " time of relaxation " being a function of the specific resistance and inductive capacity of the dielectric, and the number π. This is a typical example of the manner in which Time enters into electrical theory, and one which offers considerable scope for philosophical speculation.

The question of the time and velocity of propagation of electromagnetic phenomena is closely associated with the problem of Time, and has occupied the thoughts of many investigators, in one or other of its many forms. MICHAEL FARADAY (1791-1867), writing in 1846, remarked that a change at one end of a line of force suggests a corresponding change at the other. " The propagation of light, and therefore probably of all radiant action, occupies *time;* and, that a vibration of the line of force should account for the phaenomena of radiation, it is necessary that such vibration should occupy time also." (56) Some years later FARADAY considered this question in connexion with the force of gravity : " There is one question in relation to gravity, which, if we could ascertain or touch it, would greatly enlighten us. It is, whether gravitation requires *time*. If it did, it would show undeniably that a physical agency existed in the course of the line of force." (57) ... and again " No condition of *quality* or *polarity* has as yet been discovered in the line of static electric force; nor has any relation of *time* been established in respect of it " (p. 404)... " No relation of *time* to the lines of magnetic force has as yet been discovered. That iron requires *time* for its magnetization is well known " (p. 405)...

It has been shown by the experiments of TROWBRIDGE and DUANE (1895), SAUNDERS (1897) and MACLEAN (1899), and by the theoretical researches of GUSTAV ROBERT KIRCHHOFF (1824-1887) that the velocity of propagation of electromagnetic disturbances and light are very closely similar. This implies that the distance traversed in unit time by these two phenomena are approximately equal; an implication which led JAMES CLERK MAXWELL to formulate his electromagnetic theory of light, which regards light as an electromagnetic phenomenon.

---

(56) *Philosophical Magazine*, 3rd Ser., May 1846, p. 349.
(57) *Phil. Mag.*, June, 1852, p. 403.

This forms a typical example of the importance of Time in electrical theory. It indicates that the time of travel between two given points is the same both for electricity and light, and thus seems to infer a similarity between the two phenomena. A discussion of the consequences of such a connexion is rendered difficult by the incompleteness of the present state of electrical theory. It is manifest, however, that any discussion of this kind must take account of the underlying conception of Time, which has played such an important, though perhaps unrecognised, part in the development of this branch of science.

The Relation between Philosophical and Scientific Conceptions of Time. — Many of the conclusions arrived at in the course of philosophical speculations concerning Time seem to be at variance with, or at least very much dissociated from, the results of scientific enquiry. An examination of these discrepancies would involve a comparison of the views of the philosopher and the mathematician, which are, by their very nature of derivation, quite unamenable to any detailed reconciliation.

It becomes evident from a study of the philosophical conceptions of Time that dogmatism is not the sole domain of the mathematician. It is shared to a considerable, and perhaps less warranted, extent by the philosopher, who, to the mathematician, seems able to go on building up with great facility a structure which may rest completely on some verbose obscurity. The mathematician, on the other hand, uses a definite weapon of analysis, the accuracy and power of which is proved by the accuracy of all mathematical results and predictions in the domain of reality, as represented by terrestrial and stellar mechanics, and mathematical physics.

If mathematical analysis is able to predict exactly the phenomena of our natural domain, it seems natural to infer that its findings in connexion with phenomena essential to the scheme of this domain must be very much respected.

We must not, however, allow the mathematician to escape uncriticised. It is evident that we cannot obtain greater truth from a mathematical operation or speculation than is represented by the data upon which the operation is based. If the data do not represent actuality, then it is manifest that mathematics is

quite unable to arrive at truth. This fact represents a great difficulty in our progress towards an understanding of Time. Mathematics is concerned with physical phenomena : and the question then arises as to whether Time is a physical phenomenon. To make this assertion is to make a statement which cannot be confirmed by actual observation. We may, however, say that since all physical phenomena take place in Time, and since Time forms a necessary background, with Space, for every physical object and event, then Time may be regarded as a physical phenomenon; but only for the purpose of mathematical investigation of physical phenomena; not for speculations concerning Time itself. This view reserves for mathematics the freedom, shared by philosophy, for speculation upon the characteristics of Time as exhibited by physical phenomena.

The philosophical treatment of Time may be regarded as mainly concerned with our sense-impressions, and thus opens up an almost inexhaustible vista of speculation. An interesting example of this is afforded by the philosophy of HENRI BERGSON, as represented by his *Essai sur les données immédiates de la conscience* of 1889, wherein he attacks the conception of Time derived from the physical sciences as being too reliant upon the notion of Space. In its place is set up the idea of " la durée " which, on account of its complex derivation, is certainly not translatable into the usual idea of " duration."

In the same author's *Matière et Mémoire* of 1896 we find the suggestion that the whole of history might be contained in a very short time for a consciousness under a higher degree of tension than our own; a consciousness that would watch the development of humanity, while contracting it, as it were, into the great phases of its evolution. (58) This reflects an idealistic tendency, which calls to mind such lines as :

> A thousand ages in Thy sight
> Are like an evening gone...

It must be remembered, however, that such conceptions may not represent actuality.

The consideration of Time as some sort of extension which

(58) *Matière et Mémoire* 1896, p. 231. (English ed. p. 275).

is what Bergson endeavoured to avoid, leads directly to the question as to whether or no Time is infinite. It is held by one school of thought that we can know nothing concerning infinity, because of this condition we have no conception save the negation of a limit, and from this negation no positive conclusion can be developed. Against this doctrine it is argued by WILLIAM WHEWELL (59) that " It is not at all true that our notion of what is infinite is merely that it is *that* which has no limit. We must ask further that *what?* that space? that time? that number?..." For WHEWELL, it is necessary that the *kind* of space, surface, line, etc. be specified. Until this question is answered, we can assert nothing; not because we can assert nothing about infinites, " but because we are not told what *kind* of infinite we are talking of."

WHEWELL now passes to a criticism of the treatment given to the conception of the infinite by WERENFELS (60), from which it appears that infinites must always be regarded as equal, whereas it is WHEWELL's contention, from mathematical considerations, that " infinites may differ infinitely among themselves, both in quantity and in kind." The theory of WERENFELS is propounded in conversational form as follows : " Videtis hanc lineam *b a c*. Constituamus eam esse infinitam, et ultra terminos *b* et *c* in infinitum protendi. Dividatur haec linea in puncto *a*. Manifestum est has partes inter se esse aequales; quia utraque incipit in puncto *a*, et protenditur in infinitum. Nunc te, Daedale, rogo; hae duae partes suntne finitae an infinitae?

D. Finitae.

Ph. Ita ex duobus finitis componeretur infinitum; quod repugnat.

D. Fateor errorem. Infinitae sunt.

Ph. Jam in Scylam incidis; ita partes essent aequales toti; infinitum enim infinito aequale est. Praeterea vides utramque partem in puncto *a* terminari; non igitur finibus et terminis caret..."

This treatment of the question, in relation to Time and Space is also referred to by SHADWORTH H. HODGSON in his metaphysical essay on *Time and Space* (61), where he also concludes that infinites

(59) T*he Philosophy of Discovery*, 1860, pp. 322 *et seqq.*

(60) Quoted in MANSEL's " Bampton Lectures " for 1858, (3rd ed.) p. 301, Lect. 2., Note 15.

(61) London, 1865, pp. 135, *et seqq.*

are not necessarily equals. For this reasoning he takes the philosophical " distinction between voluntary and involuntary modes of consciousness." His conception of Time is of interest : " Time has one dimension—length. It is infinitely divisible in thought; it is infinitely extensible in thought. It admits of no minimum in division, and of no maximum in extension. For these reasons it contains every thing; nothing is short enough to slip through it, nothing long enough to outrun it. It is one in nature, for all its parts are still time. It is incompressible, for no single part can be annihilated."

The whole metaphysical outlook on the subject of Time is clearly outlined in HODGSON's book. The importance of the idea of divisibility *in infinitum* is recognised (§ 16), and the respective viewpoints of the mathematician and the philosopher defined : " The consideration of divisions as *instrumenta divisionis* of time and space, as objects expressed by points, lines and surfaces, belongs to the mathematician; that of time and space themselves and their division in consciousness, apart from the mode in which that division is expressed, is the business of the metaphysician."

The problem of Time is also of importance in connexion with psychology, wherein it presents great difficulties; so great, in fact, as to draw from LOTZE the statement that beyond some obvious considerations which lead to nothing, there is no hope of the psychologist arriving at any important result in this field. (62) Here, however, we begin to lose touch with the mathematical theory of Time, which has few connections with psychology. One such connexion is pointed out by MARY STURT in pursuing a " world time scheme " in which events are universally fixed. This gives at once a conception of Universal Time, leading to the " Newtonian Time, measured by some imaginedly perfect timepiece ticking with the heart beats of the universe." (63) The mental processes underlying our conception of temporal phenomena are also discussed. (64)

The mathematical treatment of Time is in marked contrast

(62) *Metaphysic*, II, Ch. III.

(63) *The Psychology of Time*, London, 1925, p. 143.

(64) *Ibid.*, Ch. II, V-VIII. For an account of the time sense in connexion with the occult, see THOMSON JAY HUDSON, *The Law of Psychic Phenomena*, London, 1909.

to the foregoing considerations, but it appears a debatable point whether Time is any more an abstraction for the mathematician than for the philosopher. Both tend occasionally to regard it isolated from all objective ideas. For the mathematician, Time is a variable, and receives treatment accordingly. To quote a typical example, from a work of Sir GEORGE BIDDELL AIRY, (65) (1801-1892), in connexion with partial differential equations of the second order : " Let one of the independent variables be Time..." This formal subjection of Time to the role of any ordinary variable is typical of applied mathematics. It tends to the adoption of conventions, regarding which it has been remarked (66) that " they should be made only by one who is fully conscious of their nature as conventions and does not look upon them as fixed realities beyond the power of the investigator to modify."

The fact remains that although mathematical Time is an abstraction, and as such may or may not agree with actuality, it definitely does show this agreement in all demonstrable cases. It is to be expected that around such an important conception as that of Time have sprung up numerous hypotheses and speculations, most of which have their foundation in physical principles. They often have the effect of tending to modify our conception of Time, as can be shown by a quotation from modern physics :(67) " Was NEWTON right in deliberately introducing into physics this common idea of the *flow* of time ? Surely in one great branch of physics which we owe to his genius, the mechanics of conservative systems, it has long been recognized that there is need for nothing more than the simple idea of symmetrical time, which makes no distinction between past and future." This, of course, is the modern viewpoint which identifies Time with a linear, homogeneous, mathematical continuum, agreeing with the mathematicians' view of Time as being expressible by a simple variable.

Such theories as those involved in the principle of Relativity are representative of this identification : Time becomes merely a fourth dimension, so that the proposition *post hoc ergo propter*

(65) *Partial Differential Equations*, London, 1873, (2nd ed.).

(66) *Loc. cit.*, (52). Ch. 1. § 1.

(67) GILBERT N. LEWIS, " The Symmetry of Time in Physics." *Science*, 71, p. 569, 1930.

hoc may, in ordinary language, " be no more true of a sequence of happenings in time than it is of the sequence of telegraph-poles along the Great North Road." (68) In considering this view of Time the scientist is led to regard the dimensions of the continuum, and is confronted with the problem of the actual extent of Time, with the consequent difficulties of conception and expression which the early thinkers experienced.

Scientific considerations show that the ages of the stars approximate to between five and ten millions of millions of years. Here again we experience some difficulty in adjusting our ideas of Time to the Time-scale of the universe, and we are left without any conception of Time prior to the advent of the stars, which represents the farthest retrogression which science can make.

In considerations concerning the nature of the physical world it is always the concept of Time which provides an insuperable problem, in spite of the centuries of varied studies which have penetrated deeply into all its parts. It may well appear from the findings of science as though the problem of Time must for ever remain beyond the reach of man, on account of the shortcomings of the human mind in relation to the structure of the system in which it has its being.

Whatever the answer to this question may be, the study of the problem of Time must continue always to occupy the thinking mind, and thus to form a source of advancement to our knowledge of the universe.

(Bedford, England) G. WINDRED.

INDEX OF NAMES

Airy, G. B., 20, 216.
D'Alembert, Jean-le-Rond, 19, 125.
Anderson, W., 20, 206.
Apollonius, 19, 128.
Archimedes, 19, 128.
Aristotle, 19, 126, 131-135, 142.
Arnauld, 19, 122.
Augustine, Saint, 19, 133, 134, 135.
Austin, Saint, 19, 133.
Barrow, I., 16, 126, 127, 128, 130, 132, 134, 135, 136, 137, 138, 139, 141, 142, 144, 147; 20, 194, 207.
Beck, G., 20, 206.
Bergson, H., 20, 213, 214.
Berkeley, G. Bishop, 19, 123, 124, 125, 129, 140; 20, 210.
Des Bosses, 19, 122.
Brose, H. L., 20, 203.

(68) Sir JAMES H. JEANS, *The Mysterious Universe*, Cambridge, 1930, p. 30. See also pp. 99, 119, 143 and 145, for considerations of Time in relation to physics.

Brownrigg, Bishop, 19, 126.
Buée, M., 19, 147 ; 20, 207.
Carmichael, R. D., 20, 209.
Caroline, princess of Wales, 19, 122.
Carr, H. W., 20, 199.
Cassirer, E., 19, 140, 152.
Cayley, A., 19, 149 ; 20, 197.
Challis, Reverend J., 20, 208.
Charles II, King of England, 19, 128.
Child, J. M., 19, 129.
Chrisoston, Saint, 19, 126
Clarke, S., 19, 122.
Cohn, E., 20, 203.
Collingwood, 20, 205.
Collins, J., 19, 128, 133.
Copernicus, N., 19, 122 ; 20, 196.
Diel, 19, 131.
Dirac, P. A. M., 20, 207.
Duane, 20, 211.
Eddington, A. S., 20, 195, 199, 203, 204, 210.
Einstein, A., 20, 196, 197, 198, 202, 207.
Emerson, W., 19, 125, 144.
Erdmann, J. E., 19, 123.
Eudemus, 19, 131.
Euler, L., 19, 140.
Faraday, M., 20, 211.
Fitzgerald, G. F., 20, 195.
Galileo, 19, 122 ; 20, 193
Gerhardt, C. J., 19, 123.
Goldsborough, 20, 205.
Gregory, D. F., 19, 148.
Gregory, J., 19, 121, 141.
Hamilton, W. R., 19, 147, 148, 149, 150, 151, 152, 153; 20, 197, 201, 207.
Hammond, H., 19, 126.
Herschel, J. F. W., 20, 209.
Hobbs, 19, 132.
Hodgson, S. H., 20, 214, 215.
Hudson, T. J., 20, 215.
Hume, D., 19, 140.
Jeans, J. H., 20, 210, 217.
Kant, I., 19, 137, 144, 145, 153 ; 20, 203.
Kepler, J., 19, 122.
Kirchhoff, G. R., 20, 211.
Kirkby, Reverend J., 19, 129.
Klein, F., 20, 198.
Langevin, P., 20, 199.
Leibniz, G. W., 19, 122, 123, 141, 144.
Lévy, R., 20, 206.
Lewis, G. N., 20, 205, 216.
Lorentz, H. A., 20, 194, 195, 196, 198, 202, 207.
Lotze, R. H., 20, 215.
Lucas, H., 19, 127, 128.
Lucretius, 19, 135.
Macfarlane, A., 19, 149.
Mach, E., 20, 208.
Maclaurin, C., 19, 140, 141, 142, 144.
Maclean, 20, 211.
Mansel, 20, 214.
Mapletoft, J., 19, 128.
Maxwell, J. C., 20, 195, 211.
Michelson, A. A., 20, 194, 195.
Minkowski, H., 20, 197, 198, 200, 207.
Miller, K. W., 20, 207.
Morgan, W., 19, 147.
Morley, E. W., 20, 194, 195.
Murdoch, P., 19, 141.
Napier, J., Baron, 19, 122.
Newton, I., 19, 122, 124, 126, 127, 128, 132, 133, 135, 138, 139, 141, 142, 143, 144, 145, 147 ; 20, 193, 194, 200, 216.
Peacock, G., 19, 148.
Poincaré, H., 20, 195, 196.
Poisson, S. D., 19, 13 , 136.

Pokrowski, G. I., 20, 206, 207.
Planck, M., 20, 205
Price, B., 20, 193, 194.
Ptolemy, 20, 196.
Regener, E., 20, 206.
Robb, A. A., 20, 199, 200, 201, 205.
Römer, O., 20, 204.
Rooke, 19, 126.
Saunders, 20, 211.
Schlesinger, L., 20, 203.
Schlick, M., 20, 203.
Schneider, I., 20, 203.
Schopenhauer, 19, 146.
Schultz, O., 19, 146.
Smart, W. M., 20, 209.
Smith, N. K., 19, 146, 153.
Stewart, J., 19, 143, 144.
Stone, E., 19, 129, 132.
Sturt, M., 20, 215.
Sùzuki, S., 20, 206.
Synge, J. L., 20, 205.
Theodosius, 19, 128.
Tomb, J. W., 20, 205.
Trowbridge, 20, 211.
Vasiliev, A. V., 20, 204.
Werenfels, 20, 214.
Weyl, H., 20, 203.
Whewell, W., 19, 134, 135, 136; 20, 214.
Whitehead, A. N., 20, 202, 205
Widdrington, 19, 126.
Wien, W., 20, 195.
Windred, G., 19, 147, 153.
Witte, H., 20, 203.
Wollaston, F. A., 20, 207.

G. WINDRED.

Das Alter in der Geschichte der Wissenschaft

Der Tod aus Altersschwäche als allgemeinbiologische Erscheinung. Primitive Versuche des Altersstudiums. Erforschung und Altersbekämpfung im Mittelalter. Anatomische, funktionale, biologische und biochemische Alterstheorien aus der neusten Zeit zum Zweck der Aufklärung des Altersmechanismus. Zusammenfassung.

Das Altersproblem muss als das grundlegende Problem der Biologie angesehen werden, und zwar einerseits darum, weil das Alter eine allgemeinbiologische mit allen Lebensfunktionen in Verbindung stehende Erscheinung darstellt, anderseits, weil die philosophische Bedeutung dieses Problems zur Möglichkeit einer Lebensverlängerung führt, die sich nicht nur auf einzelne Individuen erstreckt, sondern auch auf ganze Gesellschaften von Personen, die in sich den intellektuellen Schöpferwert der Menschheit darstellen.

Der Tod aus Altersschwäche kennzeichnet sich dadurch, dass seine Entstehungsbedingungen in der eigentlichen inneren Organisation des Gewebes, der Zelle oder des Organismus liegen und der eintretende Tod schon aus diesem Grunde den Charakter der Natürlichkeit in sich trägt. Da der natürliche Tod, der Tod aus Altersschwäche bei allen Lebewesen zu beobachten ist, so erklärt sich hierdurch sein allgemeinbiologischer Charakter. Diese allgemeinbiologische Eigenschaft der Erscheinung des natürlichen Todes ist denn auch der Grund dafür, warum wir uns vom Standpunkt der allgemeinen Thanatologie mit ihr zuerst beschäftigen.

Infolge der Vielseitigkeit der natürlichen Todeserscheinungen, die ihrerseits in den vielseitigen Lebensformen des Tier- und Pflanzenreiches begründet liegen, hat sich eine reiche Anzahl von Theorien und Hypothesen herausgebildet, die den Versuch machen, eine Erklärung für den natürlichen Tod zu liefern. Der Umfang vorliegender Arbeit erlaubt es uns nicht, eine bis

ins Einzelne gehende Betrachtung der verschiedenen Anschauungen über den Mechanismus des natürlichen Todes zu geben, wir halten uns jedoch vom Standpunkte der allgemeinen Thanatologie aus verpflichtet, wenigstens in kurzen Umrissen ein Bild von der geschichtlichen Vergangenheit der Frage zu geben und die am meisten kennzeichnenden Züge aus der Phänomenologie des Alters zu unterstreichen.

Der erste Versuch eines Herantritts an das Altersproblem gehört den Griechen HIPPOKRATES (460-370, v. Chr.) und GALEN (131, v. Chr.), die den Grund des Alterns in einer Veränderung, bezw. einem Verderben des Blutes sahen. Spätere Gelehrte verglichen (wie HALLER u.a.) das Alter mit Abnützung. Nach Seneca ist „senectus ipse morbus".

Im Mittelalter wurde das Alter nur im Rahmen eines Suchens nach Mitteln und Methoden einer unmittelbaren Verjüngung studiert, und alle Versuche in dieser Richtung kennzeichnen sich durch das Vorherrschen des dunkelsten Aberglaubens: Suchen nach dem Stein der Weisen, dem Lebenselixier und dem Jungbrunnen.

Erst in der Mitte des vergangenen Jahrhunderts, als der Grund zur physiologischen Wissenschaft gelegt wurde, hat man die ersten Versuche gemacht, die Erscheinung des Alterns als des natürlichen Todes wissenschaftlich zu analysieren.

1853 hat RÉVEILLÉ-PARISE den Gedanken ausgesprochen, dass das Ableben eines menschlichen Organismus durch eine greisenhafte Degeneration in den Lungen hervorgerufen werde. Als Resultat dieser Degeneration wird eine unzureichende Versäuerung des Blutes erzielt, und hierdurch eine Veränderung seiner Zusammensetzung, die den Ernährungsprozess der Gewebe stört, sodass auch diese degenerieren.

MOREL (1857) hält das Altern für das Resultat einer Vergiftung des Organismus, die als Folgeerscheinung eines regelmässigen, wenn auch nicht unmässigen Verbrauchs von betäubenden und aufreizenden Mitteln auftritt.

1865 schrieb CLAUDE BERNARD betreffs der Entstehungsbedingungen des natürlichen Todes: Die Zusammensetzung des Organismus ist das wirklich vitale. Leben ist Schöpfung, vergleichbar einem grossen chemischen Zersetzungs- und Aufbauprozesse; dies sind die wirklichen Todeserscheinungen, wenn sie am Lebewesen vor sich gehen. Und, — ein Ding, das wichtig

zu bemerken ist —, wir fallen hierbei einer Gewohnheitseinbildung zum Opfer, da wir, indem wir die Lebenserscheinungen kennzeichnen wollen, in Wirklichkeit über Todesvorgänge sprechen.

1881 entwickelt MARTIN die Anschauung, dass das Altern durch Arteriosklerose hervorgerufen wird, und zwar nicht durch Periarteritis, sondern durch Endoarteritis obliterans, die zu einer Verengerung der kleinen Arterien führt, sodass die normale Ernährung der Gewebe unterbunden und distrophische Sklerose verursacht wird.

1882 spricht sich BÜTSCHLI dahingehend aus, dass der tödliche Ausgang greisenhafter Gebrechlichkeit durch Verausgabung der im Blute befindlichen Fermentstoffe veranlasst werde, und dass, da diese sich nicht wieder auffüllen liessen, eine unaufhaltbare Veränderung in der Zusammensetzung sowohl des Blutes, wie auch der Gewebe vor sich gehe.

Nach HAMELIN ist die Veränderung in der Atmungsfunktion, durch die das Ableben bedingt wird, eine zweitgradige Erscheinung. Sie werde hervorgerufen durch eine im Alter auftretende Verknöcherung der Rippenknorpel.

Nach Ansicht anderer Ärzte desselben Zeitabschnittes ist das Alter eine Folge von Arteriosklerose. Nach der Auffassung DEMANGES (1886) und anderer ruft die Arteriosklerose eine Sklerose der inneren Organe hervor, die allmählich allgemein wird und zu weiterem Auswachsen der Bindegewebe im Organismus führt.

Alle Theorien, die das Altern durch Arteriosklerose erklären, nehmen im Einverständnis mit CAZALIS Auffasung: *on a l'âge de ses artères* (der Mensch ist so alt wie seine Arterien) ihrerseits als Grund der Arteriosklerose eine schädliche Einwirkung des verdorbenen Blutes auf die Gefässe an; sie sind hierin ebenso folgerichtig wie jene obenerwähnten Theorien, die das Altern durch Veränderung der Atmungsorgane erklären.

1886 erklärt BROUSSE, indem er von der Arteriosklerose spricht, dass sie die Ernährung der parenchimen Organe stört, er hält jedoch die Arteriosklerose nur für ein Alterssymptom, nicht für einen Altersgrund. Zur Erklärung des Alters zieht er BOUCHARDS Lehre heran, die von Intoxicationswirkungen und Stoffwechselsverlangsamungen im Organismus spricht.

Nach PASCHUTIN (1885) kennzeichnet sich das Reagieren des

Protoplasma auf äussere Einflüsse vor allen Dingen darin, dass sich verschiedene Kräfte (Wärme, Elektrizität, mechanische Arbeit u. ähnl.) aus ihm freimachen und hierbei ein Teil seiner Materie unvermeidlicherweise in einfachere Verbindungen zerfällt. Hieraus folgt, dass *die organisierte Materie funktionierend untergeht.*

KANSCHTAT betrachtet das Alter als greisenhafte Involution.

1883 stellte ALEXANDER GOETHE die Theorie auf, dass der natürliche Tod mit Vorgängen auf dem Gebiete der geschlechtlichen Vermehrung in enger Verbindung stehe, und dass der Organismus der Erzeuger sich erschöpfe und somit das Altern bedinge. Später hat HANSEMANN diese Ansicht dahingehend abgeändert, dass der physiologische Tod eine Folge von Atrophie der Geschlechtsdrüsen und völliger Vernichtung des Keimplasma sei.

Das Altern der Infusorien spricht sich nach der Meinung MAUPAS (1888) in folgenden Erscheinungen aus: Grössenverminderung des Organismus, Degeneration des Kernapparates und der Mundöffnung, sowie physiologische Degeneration, die zum Tode fürht.

1888 versuchte MALY die Altersursache mit Erscheinungen von Darmintoxikation in Verbindung zu bringen. Diese Anschauung wurde später von METSCHNIKOFF weiterentwickelt. Die Anhänger dieser Theorie nehmen an, dass das Klimakterium die Entwicklung des Organismus hemmt und demzufolge Heteroplasie eintritt. (DURAND-FARDEL.)

BROWN-SEQUARD schreibt 1889 ganz einfach: „Die Erscheinung des Verwelkens eines Organismus ist das Resultat einer Funktionsverminderung in den Geschlechtsdrüsen".

GÖFFE stützt sich auf die bekannte Tatsache, dass einige Insekten nach dem Begattungsakte sterben, und andere, die ihn vermieden haben, bis zum nächsten Frühjahr leben bleiben. Er nimmt an, dass Alter und natürlicher Tod eine Folge des Geschlechtslebens sind.

1889 bezeichnet SEIDEL den Prozess der Abnützung des Organismus „senescens" als eine unvermeidliche und unheilbare Krankheit.

Nach der Theorie KASSOVITZ' wird ebenfalls das Altern durch eine Verminderung der assimilatoren Prozesse bedingt, nur erscheint diese Verminderung als Resultat einer ständigen Ver-

mehrung besonderer metabolischer Stoffwechselprodukte, Metaplasmen, die sich auf dem Wege eines passiven chemischen Zerfalls der protoplasmatischen Substanz herausbilden und sich in ein inertes, reizfestes Gewebe verwandeln. Zu den Metaplasmen haben wir zu rechnen: die Produkte der regressiven Metamorphose, den Faserknorpel und schliesslich die homogene Zwischenzellensubstanz, die den Raum im Protoplasmanetz anfüllt. Im Laufe der Jahre verändert sich das Verhältnis zwischen Protoplasma uns Metaplasma zu Gunsten des letzten und das Protoplasma, der Träger des Stoffwechsels, verschiebt sich nach innen, indem es so den Beginn des Alterns herbeiruft.

1891 stützt sich SABATIER bei Erklärung des Alterns auf die Zellendifferenzierung, nimmt aber an, dass durch diese ein Ableben des Organismus aus dem Grunde veranlasst wird, weil das primitive protoplasmatische Element sich allmählich erschöpft und abstirbt. Demzufolge ist nach SABATIER das Altern eine indirekte Folge der Zellendifferenzierung.

WEISSMANN (1892) löst die Frage vom Altern und vom Tode durch ein Studium dieser Vorgänge bei den Einzelligen. Nach seiner Meinung fehlt bei ihnen die Erscheinung des Gebrechlichwerdens, da infolge der Zweiteilung das Muttertier nicht verschwindet, sondern mit allen seinen Funktionen in die Tochterzellen übergeht, weshalb bei ihnen der natürliche Tod nicht eintritt. Bei den Vielzelligen finden sich die sogenannten somatischen und keim-idioplasmischen Zellen, die in besonderen Geschlechtsdrüsen ihr Vorkommen haben. Wenn dieses Soma überflüssig wird, verschwindet es allmählich und führt so zum Tode. Der Organismus, der diese Keimzellen ausgeschieden hat, erfüllt hierdurch seine Pflicht der Art gegenüber und wird im weiteren völlig überflüssig. Von diesem Standpunkte aus können wir Alter und Tod als eine besondere Vorrichtung des Organismus im Kampfe um das Dasein betrachten. WEISSMANN sagt, dass die Lebensdauer eines Körpers durch die natürliche Zuchtwahl so reguliert ist, dass jede Art genau über jene Lebensdauer verfügt, die für sie nach ihren physischen Eigenschaften, ihrer physiologischen Arbeitsfähigkeit und ihren angepassten Lebensbedingungen am vorteilhaftesten ist.

Nach HODGE (1894-1895) sind die Pigmentkörnchen im

Zentralnervenorgan für das Problem des Alterns von grösster Bedeutung.

Boy-Teissier (1895) nimmt an, dass durch Differenzierung die Lebensfähigkeit der Zellen herabgemindert wird, dass darum die Lebensdauer der Organe allmählich sinkt und ihre parenchimatösen Elemente atrophiert werden, wobei proportional dem Altersgrade das Bindegewebe auswächst.

Delbœuf (1896) geht von der Tatsache aus, dass die Zelle, je höher sie differenziert ist, desto geringere Lebensfähigkeit besitzt, und nimmt an, dass das Altern des Organismus eine direkte Folge der Zellendifferenzierung ist.

Nach Pawlinoffs (1896) Ansicht wird während der Differenzierungsperiode, zur Zeit der vollen Kraftblüte, der menschliche Organismus schon weniger intensiv mit Sauerstoff versehen, als während der Wachstumsperiode. Während des Altersprozesses verringert sich die Sauerstoffversorgung der Zellen des menschlichen Organismus noch mehr, und als Folgeerscheinung treten die verschiedenen atrophischen Vorgänge auf. In einem gewissen Grade ist bei Sauerstoffmangel hinreichende Ernährung nicht denkbar, sie nimmt immer mehr ab, die Zellentätigkeit geht zurück, schliesslich tritt Atrophie und Degeneration ein, die ein Weiterleben der Zelle selbst unmöglich machen.

1898 ist Jukovsky und später Kulagin (1899) an Hand ihrer Versuche an Paramezium zu der Annahme gelangt, dass Degenerationserscheinungen in alten Kulturen als Folge von Anhäufung schädlicher Ausscheidungen im organischen Körper auftreten.

Marinesco (1900) versucht, das Zellenalter nicht durch den intimen Kampf der Nervenelemente, sondern durch Mangel an chemischer Synthese seitens der Nervenzelle zu erklären.

Nach Hermann Nothnagel (1900) wird im biologischen Sinne der Absterbeprozess im Laufe weniger Stunden abgeschlossen. Der Tod des Menschen beginnt beim Herzen. Mit dem letzten Herzschlage endet im biologischen Sinne das Leben des Menschen oder Tieres.

Bart stellt eine Theorie auf, in welcher er erklärt, dass die Hauptursache des Todes, des pathologischen sowohl wie des natürlichen, in einer Vergiftung der Zellen durch die Umgebung besteht. Hierdurch wird nach Barts Worten der grundlegende Lebensakt — Ernährung und Assimilation — unterbunden.

Auf diese Weise ist der Tod immer das Resultat einer Unterbrechung der Zellenernährung, sei es weil das Protoplasma assimilations- und desassimilationsunfähig wird, oder sei es, weil die Umgebung, in der sich die betreffenden Zellen befinden, Veränderungen unterworfen wird, die den Stoffwechsel unmöglich machen.

1900 gründete MÜHLMANN seine Theorie, als deren Ausgangspunkt das WEISSMANNSCHE Gesetz von der Unsterblichkeit der Einzelligen diente. Die Ursache dieser Unsterblichkeit liegt in der morphologischen Verschiedenheit der ein- und vielzelligen Organismen. Infolge der symbiotischen Erscheinungen bei den Vielzelligen entstehen: 1° das Wachstum, das nicht etwa als ein Vergrösserungsprozess des Zellenumfangs (wie bei den Leberzellen) anzusehen ist, sondern als Proliferation, in deren Folge: 2° Alterserscheinungen, Assimilationsverminderung und Tod eintreten. Die Assimilationsverminderung hat ihren Grund darin, dass die Zellenvermehrung nicht parallel mit dem Wachstum ihrer Ernährungsanpassung vor sich geht.

Der Autor führt weiter aus, dass der Grund des allgemeinen organischen Todes abhängt: 1° von der Körperoberfläche und der grösseren oder geringeren Entfernung seiner einzelnen Teile von ihm, und 2° von den Wachstumsgesetzen der lebenden Materie. Dies spricht sich zuerst in der Degeneration der Nervensystem-Zentren aus. In der Jugend sind die Ganglienzellen imstande diese zentralen Funktionen auszugeben, im Alter jedoch wird dank der in ihnen vor sich gehenden degenerativen Prozesse diese Kompensation unmöglich und der unvermeidliche Tod tritt ein.

KÖVÖSI (1901) sieht in den Störungen des intercellularen Stoffwechsels einen Grund für die Involutions-Kachexie.

Einer dieser Gründe, und zwar einer der hauptsächlichsten, meint FINOT (1901), ist in der Todesfurcht zu suchen. Der Mensch, der eine bestimmte Altersstufe, oder auch nur einen bestimmten Seelenzustand erreicht hat, wird einer eigenartigen Autosuggestion bezüglich der Todesfrage unterworfen. Er stellt sich dann vor, dass er seinem Ende nahe sei, und die Furcht vor dem Tode wird dann in demselben Grade ein Nährstoff für ihn, wie die lebenerhaltende Speise. Von diesem Augenblicke beginnt der Todesgedanke ihn gleichsam in seinem Bann zu

halten. Überall und fortwährend scheint er ihn zu rufen. Die philosophische wohltätige Hoffnung auf ein jenseitiges Leben weicht einer kleinmütigen Angst vor dem Verlust des Lebens. Und der Mensch nährt sich von dieser Angst, vergiftet sich mit ihr und stirbt an ihr.

FRIEDMANN (1902) sagt, man könne beweisen, dass der Kreis der Lebensvorgänge mit der Zeit immer enger und enger werde, und zwar unter dem Einfluss der Volumverminderung und des Funktionsrückganges des Körpers und unter dem Einfluss einer allmählichen Beschleunigung des Blutumlaufes und der Atmungsfunktion, sowie einer Temperatursteigerung, obwohl einzelne Organe bis ins tiefe Alter nicht zu wachsen aufhören. Es unterliegt gewiss keinem Zweifel, dass man sich noch engere Grenzen vorstellen könnte, in denen menschliches Leben möglich wäre. Letzten Endes würde der Tod als Folgeerscheinung eines bis auf das äusserste Minimum zurückgeführten Stoffwechsels eintreten.

JENNINGS (1902) widmet bei seinen Beobachtungen an den Einfachsten besondere Aufmerksamkeit der Konjugationserscheinung und ihren Folgen für Altern und Tod der Einzelligen.

JICKELI (1902) sieht gerade im unzulänglichen Stoffwechsel und der Anhäufung von Zerfallprodukten den Grund des Alterns.

CALKINS hat 1902-1904 auf experimentalem Wege den Versuch gemacht, die Depression der Einzelligen als Alterserscheinung zu erklären.

Nach ENRIQUES (1903) Meinung erklärt sich die in MAUPAS und CALKINS Versuchen beobachtete Degeneration und greisenhafte Ausartung in der Hauptsache durch bakteriale Intoxikation.

RUBNER (1904) hält das Aufhören des Wachstums für die Grundursache von Alter und Tod. Er sagt: Das Leben endet bei den Tieren dann, wenn die Gewichteinheit dieselbe Menge von Energie verbraucht und umgesetzt hat. Der Tod ist eine Konstitutionserscheinung. Nur dem Menschen ist eine Lebensmasse geworden, welche ihm ein viel längeres Leben sichert, als man nach seinem sonstigen Verhalten vermuten sollte.

BUHLER (1904) legt mit Recht, sagt KORSCHELT, ein grosses Gewicht auf die Stoffwechselvorgänge und betrachtet das Problem des Alters und des Todes vom chemisch-physikalischen Standpunkt. Danach ist eine Assimilation nur möglich, solange

zwischen der lebenden Substanz und den eingeführten Stoffen Affinitäten, d.h. ausgleichbare elektrochemische Differenzen bestehen.

Nach der METSCHNIKOFFschen Theorie (1904) ist vorzeitiges Altern das Resultat einer langsamen Darmautointoxikation. Über ein anderes wichtiges Symptom des vorzeitigen Alterns, die greisenhafte Atrophie, sagt M. folgendes: Atrophische Erscheinungen herrschen im Alter vor. Bei physiologischer sowohl wie bei pathologischer Atrophie geht zwischen den Zellen einerseits ein Überfall von Makrophagen und andererseits Zellenselbstschutz durch Ausscheidungen vor. Greisenhafte Atrophie führt sich auf Atrophierung der edleren spezifischen Gewebeelemente und ihren Ersatz durch hypertrophiertes Bindegewebe zurück. Überhaupt besteht die greisenhafte Ausartung vor allem in einer Art von Makrophagitolyse, die den Untergang der edleren, infolge von Darm-Autointoxikation nicht eines genügenden Selbstschutzes fähigen Elemente hervorruft. Zuerst geht eine Schwächung der Zellen vor sich, und dann erst ihr Verschlingen und Ersticken durch Phagozythen.

Nach E. SCHULTZE (1905) hat die Natur alle Mittel besessen, um das Individuum unsterblich zu machen, aber sie hat ihm den Tod auserlesen. Anstelle einer ständigen Verjüngung der einzelnen Organe mit Hilfe einer Verjüngung ihrer Zellen hat sie die Verjüngung des ganzen Organismus mit Hilfe einer einzigen Zelle gewählt. Sie hat uns die Unsterblichkeit genommen und gab uns zum Tausch dafür die Liebe.

HERTWIG (1906) schreibt, dass die Einzelligen zuweilen bei intensiver Ernährung und nach Vermehrungsprozessen in Depressionszustände verfallen und zu dieser Zeit ihre Funktionen hemmen oder einschränken, um die Zellenbestandteile wieder aufzubauen. Bei den Zellen des menschlichen Körpers ist etwas derartiges weniger möglich, da nicht eine einzige Funktion unterbrochen werden kann. Wenn die Zellen, die diese Funktionen ausüben, infolge zeitweilig übermässiger oder allzu lange anhaltender Tätigkeit nicht mehr imstande sind, sich in ausreichender Weise zu erholen, so kommen sie in Verfall und gehen schliesslich zu Grunde. Demnach führt die höchste Differenzierung vom Zellenteiltode zum Tode des Ganzen.

Nach TEICHMANN (1906) stellt der Tod an sich keine innere

Notwendigkeit dar, und ist kein wesentlicher Charakterzug der lebenden Materie.

Die von POPOFF (1907) an vielen Infusorien vorgenommenen Messungen berechtigten ihn zu dem Schlusse, dass die der Aufnahme unterliegenden Produkte im Plasma angehäuft werden. Hierbei empfinden die Zellen Depression, sie altern und müssen im Endresultate, falls keine Konjugation eintritt, untergehen.

Nach TRANJEN (1908) entsteht das Alter durch Autointoxikation mit Stoffwechselprodukten als Folge schädlicher äusserer Einflüsse.

Nach HEIG (1908) müssen zu den Faktoren, die wenn auch nur indirekt die Annäherung des Alters fördern, alle diejenigen Produkte gerechnet werden, deren Verbrauch eine übermässige Anhäufung von exogener Harnsäure im Organismus hervorruft. In der experimentalen Pathologie gilt die Tatsache als sicher festgestellt, dass die Einführung von Harnsäure in jeder Form in den Organismus die Blutzirkulation nicht nur im Hirn sondern im ganzen Körper verlangsamt und somit seine Lebenstätigkeit herabsetzt. Als Folgeerscheinung reichlichen Fleischgenusses treten Antointoxikationszustände und chronische Erschöpfung ein, die schliesslich in Alter und Tod übergehen.

LÖB (1908) führt aus: es erscheine unwahrscheinlich, dass ein so feiner Mechanismus wie das Herz mit solcher Regelmässigkeit während der ganzen menschlichen Lebensdauer funktionieren könne. Das Geheimnis, das über dieser Erscheinung waltet, würde leicht aufzuklären sein, wenn wit mit Sicherheit behaupten könnten, dass alle Zellen tatsächlich unsterblich sind. Dann hätten wir nicht mehr die lange Tätigkeitsdauer des Organismus, sondern das Aufhören dieser Tätigkeit bei Todeseintritt zu erklären. Auf diesem Wege sehen wir, dass die Idee von der Unsterblichkeit der somatischen Zellen in verallgemeinerter Auffassung eine der wichtigsten Grundlagen für die volle physikochemische Analyse der Lebenserscheinungen werden könnte, da nur mit ihrer Hilfe die lange Lebensdauer des Organismus verständlich wird.

Das Problem des Todes und der Zerstörung des Organismus stellt sich nach LÖB folgendermassen dar: 1° warum tritt beim Aufhören der Atmung sofortiger Tod des gesamten Körpers ein, oder anders gesagt, warum beginnt der Körper zu zerfallen? 2° Was schützt bei normalem Atmungsprozess den Körper vor

dem Verfall, trotzdem seine Temperatur und Feuchtigkeit eine Zersetzung begünstigen?

„Von den Zellen unseres Körpers, sagt RIBBERT (1908), kommen für den tödlichen Ausgang direkt nur die Ganglienzellen in Betracht. Der natürliche Tod ist ein Gehirntod. Die schliesslich hochgradige Atrophie der Ganglienzellen ist mit dem Leben nicht mehr vereinbar. Die Veränderungen der Zellen aller anderen Organe befördert die allgemeine Abnahme der Lebensenergie und damit indirekt die Erlahmung der Ganglienzellen."

Nach D. VON HANSEMANN (1909) stehen die Prinzipien der Selektion und des physiologischen Todes, als Folgeerscheinung der Existenz von Sexualzellen, im engsten Zusammenhange miteinander und stützen sich gegenseitig, sodass man bei Verneinung des einen auch das andere ablehnen müsste.

EWALD (1909) versucht eine endokrinologische Erklärung des Alterns zu liefern.

BLOCH setzt die Existenz eines natürlichen Todes beim Menschen in Zweifel.

Nach ROESER (1910) ist das Altern das Resultat von gegenseitiger Aufeinanderwirkung innerer (primitiver) und äusserer (zweitgradiger) Lebensfaktoren.

Wenn die Drüsen, sagt LORAND (1911), im Alter verschiedene degenerative Veränderungen durchmachen, so hemmen sie hierbei die Ausscheidung giftiger Stoffe aus dem Organismus. Demzufolge sind wir berechtigt, das Alter als einen chronischen Autointoxikationszustand anzusehen. Das versteht sich auch für andere Stoffwechselkrankheiten, wie Myxoedem, Morbus Basedowi und Diabetes. Aber wenn die Veränderungen in den Organen der inneren Sekretion auch das Bild eines vorzeitigen Alterns hervorrufen, sind wir denn dadurch in stand gesetzt zu beweisen, dass diese Veränderungen auch das normale physiologische Alter verursachen? LORAND glaubt, dass zwischen diesen beiden Zuständen nur ein quantitativer Unterschied besteht. Dieselben Schädigungen, die beim 30jährigen Menschen alle Anzeichen vorzeitigen Alterns hervorrufen, können bei einem andern dank hygienischer Massnahmen den Organismus bis zu 60-70 Jahren erhalten. Aber gleichwohl treten diese Schädigungen ein, und niemand unter den auf der Erde Lebenden kann ihnen entgehen.

WOODRUFF (1911) und METALNIKOFF (1919) haben Massenversuche an Vielzelligen vorgenommen, die zu folgenden Resultaten führten: 1° Die physiologische Zellendegeneration der Einfachsten, die sie zum Untergang führt, wird (natürlich unter Laboratoriumsverhältnissen) hervorgerufen durch Anhäufung von Produkten der eigenen Lebenstätigkeit der Zelle in der Umgebung, die sie bewohnt; 2° Diese Anhäufung von Zerfallprodukten hemmt die Assimilationsphase des Stoffwechsels; 3° Konjugation ist zur Beseitigung physiologischer Degeneration nicht unbedingt notwendig; 4° Unter natürlichen Lebensbedingungen, wo keine Anhäufung von Dissimilationsprodukten vorfällt, gibt es auch kein Alter.

HANS (1912) misst dem Gehirn im Prozess der Altersentstehung grosse Wichtigkeit bei.

DASTRE (1912) sagt: Wir altern und sterben, und sehen alle Lebewesen, die uns umgeben, ebenfalls altern und verschwinden. Wir finden keine Ausnahme von diesem erbarmungslosen Gesetz und betrachten es als eine natürliche Unabwendbarkeit. Aber ist diese Verallgemeinerung auch begründet? Oder gibt es im Gegenteil unsterbliche Wesen?

HARMS (1913) bringt die Altersveränderungen mit den Veränderungen in den Ganglienzellen in engen Zusammenhang.

Nach der Anschauung BERGSONS (1913) stellt die lebende Materie ein organisches System dar, das sich wie eine Lawine auswächst, in dem es keine Wiederholungen gibt, in dem ein jeder Vorwärtsschritt eine Neuschöpfung, einen Freiheitsbeweis bedeutet. Demnach ist der lebende Organismus eine schaffende Existenz. Im Gegensatze hierzu ist die unorganische Materie eine nicht schaffende sondern zerfallende *(qui se défait)* Existenz. In ihr gibt es keine Freiheit, kein Schöpfertum, es herrscht nur die Notwendigkeit. Und so festgefügt, dauerhaft und unveränderlich uns die unorganische Welt auch erscheinen mag, in Wirklichkeit zerfällt sie und zerstört sie sich allmählich und unweigerlich. Der lebende Organismus dagegen, der sich ständig verändert, nie derselbe bleibt, von der geringsten Veränderung seiner Umgebung zu leiden hat, der von anderen Organismen vernichtet wird, der da stirbt und sich wiedergebiert, zeichnet sich durch ungewöhnliche Dauerhaftigkeit und Lebensfähigkeit aus.

Wenn man, sagt CARREL (1913), die Konzentration der Plasma-

proteine verringerte, so würden die Zellen, in eine weniger hemmende Flüssigkeit gebadet als ihre gewöhnliche Umgebung, ihre Tätigkeit verstärken. Sie würden vielleicht imstande sein, ein aus weniger hemmenden Substanzen zusammengesetztes Plasma zu regenerieren. In diesem Falle würde der Prozess des Alterns bis zu einem gewissen Grade rückkehrbar werden. Dieses wurde einer experimentalen Nachprüfung unterworfen.

TRETJAKOFF (1913) sagt, dass uns bisher keinerlei Theorien von einer unsterblichen Existenzmöglichkeit des Körpers haben veranlassen können, auf die Meinung zu verzichten, dass der Tod der unausbleibliche Begleiter des Lebens ist. Wir müssen uns mit ihm abfinden, wie wir uns mit dem Gewicht einer Bürde abfinden.

FALTA (1913) erklärt das Altern als eine Folge der Tätigkeitsunterbrechung aller endokrinen Drüsen.

MINOT (1913) betrachtet das Altern als eine Verringerung der Wachstumsintensität, die er in jedem gegebenen Falle an dem absoluten Körpergewicht abmisst. Diese Verringerung der Wachstumsintensität bringt er in kausalen Zusammenhang mit der zythomorphosen Zellendifferenzierung. In seiner Arbeit „Moderne Probleme der Biologie" schreibt dieser Forscher: „zwischen Differenzierungsgrad und Wachstumsintensität besteht ein kausaler Zusammenhang; das wird durch die folgenden unmittelbaren Beobachtungen bestätigt: nichtdifferenzierte Zellen teilen sich schnell, die differenzierteren langsamer und die völlig differenzierten überhaupt nicht. Diese Erwägungen haben mich zu der Folgerung geführt, dass wir die Differenzierung als den wesentlichen Grund des Alterns anzusehen haben. Die Differenzierung geht in der Hauptsache im Protoplasma und zwar infolge des Wachstums vor sich; demnach wird also das Altern durch Vermehrung und Differenzierung des Protoplasma herbeigeführt."

Nach SCHÄFER (1913) erscheint der Tod als Vollendung des Lebenszyklus.

„Der menschliche Organismus, schreibt ERATO-SLUTZKY (1913), kann der Einfachheit halber als ein Mechanismus angesehen werden, in welchem physiko-chemische Prozesse vor sich gehen, die während ihres Verlaufes unter ständiger Kontrolle der Hirnzentren stehen. Jeder Mechanismus muss im Laufe der Zeit und mit dem Masse der geleisteten Arbeit altern und unbrauchbar

werden. Der frühere oder spätere Eintritt der Unbrauchbarkeit hängt vom Arbeitstempo, von der Quantität und dem Charakter seiner Arbeit ab. Am ehesten wird ein Organismus unbrauchbar, wenn die Arbeit dem Plane seines mechanischen Aufbaus und seiner Bestimmung nicht entspricht. Von diesem mechanischen Standpunkte aus kann behauptet werden, dass bestimmte physiologische Verhältnisse des Organismus von der Ernährung abhängen und seine vorzeitige Abnutzung, d.h. verfrühtes Altern bedingen."

Prof. W. W. SAVJALOFF versucht die Altersentstehung folgendermassen zu erklären: „Es ist bekannt, so schreibt er, dass der Untergang von Bakterienkulturen nicht durch Erschöpfung der Nahrungselemente ihrer Umgebung erklärt werden kann. Kolonien gehen unter infolge Vergiftung durch die Produkte ihrer eigenen Lebenstätigkeit, und es scheint als ob diese Selbstvergiftung durch Stoffwechselerzeugnisse tatsächlich eine grundlegende Tatsache von allgemeiner Bedeutung ist. Der menschliche Körper hat grosse Ähnlichkeit mit einer solchen Kultur lebender Materie: in einem Hautsack sind ungefähr zweiundeinhalb Eimer Nährflüssigkeit enthalten (Blut und Lymphe), auf dieser Nährflüssigkeit vegetieren die Lebenselemente des Körpers, — seine Gewebezellen. Es ist klar, dass diese Kultur sich mit der Zeit verschmutzen muss und eine Umimpfung erfordert; der Wechsel der Generationen stellt an sich nichts anderes dar, als eine solche Umimpfung des lebenden Stoffes auf ein neues Substrat. Von diesem Standpunkte aus wird die Unsterblichkeit der einzelligen Tiere und die Sterblichkeit der vielzelligen verständlich: die Organisation eines vielzelligen Körpers erschwert in unvermeidlicher Weise den Stoffwechsel mit der Umgebung und die Ausscheidung giftiger Zerfallprodukte; die im engen Raume innerhalb der Körperhüllen konzentrierten lebenden Zellen vergiften sich durch ihre Abfälle, wie die Bevölkerung einer mittelalterlichen, in den engen Kreis ihrer Mauern eingezwängten Stadt."

„Obwohl bei den einzelligen Lebewesen, sagt Prof. KASCHTSCHENKO, der Mutterleib in den Leib der Nachkommenschaft übergehen kann, so kann dies doch nicht unbegrenzt fortgesetzt werden." Beobachtungen an Gregarinen, Radiolarien, einfachsten Fadentierchen und Infusorien haben gezeigt, dass von Zeit zu

Zeit der Überschuss von festen Differenzierungsprodukten abstirbt, vernichtet oder in Form von Nekron abgestossen wird (Karionekron, Plasmonekron, Zithonekron), dessen Bildung für die Lebenstätigkeit der nächsten Generation notwendig ist. In dieser Notwendigkeit von Nekronbildung besteht denn auch der Ausgangspunkt für die Entstehung des natürlichen Todes. Das Nekron kann man als Leichenkeim bezeichnen, sein Bildungsprozess ist der Beginn von Alter und Tod."

Nach FRIEDENTHAL (1914) steht die Schnelligkeit des Alterns mit den Faktoren der „Zephalisation" in enger Verbindung.

SCHEPOTJEFF (1914) zieht auf Grund der chemischen Vererbungstheorie den Schluss, dass, wenn man eine Kontinuierlichkeit des vererbten Stoffes nicht annehmen dürfe, man ebensowenig an die Unsterblichkeit der Einfachsten, wie WEISSMANN sie definiert, glauben könne. Den Grund des Absterbens ganzer Gruppen (Senilität) habe man zu suchen in der allmählichen Widerstandsschwächung biologischer Strukturen während des gesamten phylogenetischen Entwicklungsganges der Art.

A. LIPSCHÜTZ (1915) definiert den Tod als „irreversiblen Stillstand des Stoffwechsels der lebendigen Substanz". Die Grundfrage, ob es einen Tod gibt, der mit Notwendigkeit aus den Lebensbedingungen erwächst, „einen Tod, der zum Leben gehört", wird in dem WEISSMANNschen Sinne beantwortet. Für die Einzelligen ist der Tod aus Altersschwäche keine Notwendigkeit, bei den Metazoen dagegen hat er sich entwickelt.

„Der Tod ist nur das letzte Endglied langer Entwicklungen, — sagt VERWORN (1915) —, das Problem des Todes und das Problem der Entwicklung enthält dieselbe Frage, die Frage, warum sich die lebendige Substanz während ihres individuellen Daseins fortdauernd verändert. Erst ein tieferes Eindringen in den Chemismus der lebendigen Zelle wird imstande sein, die besonderen Ursachen für diese Erscheinung aufzudecken."

SCHLEIP (1915) sieht den Beweis für die Anschauung, dass jedes Tier aus rein inneren Gründen nach einer bestimmten Zeit sterben muss, in der Tatsache des Vorhandenseins einer oberen Grenze der Lebensdauer, die auch unter den günstigsten Bedingungen nicht überschritten werden kann.

Nach CHILD (1915) steht das Altern mit einer Herabminderung der Anpassungsfähigkeit und Regenerationsfähigkeit im Zusam-

menhang, was er in der Hauptsache durch Versuche an Planarien belegt. „Soweit sie ihre Aufmerksamkeit dem Altern zugewandt haben, sind Anatomen, Histologen und Pathologen oft nicht imstande gewesen das zu erkennen, was uns das Studium der niederen Organismen als Tatsache anzuerkennen zwingt, nämlich, dass das Alter nur ein Entwicklungsaspekt ist, sondern sie haben ihre Aufmerksamkeit gerichtet und ihre Theorien aufgebaut auf die morphologischen Veränderungen, die im späteren Leben vorkommen und insbesondere in jener Epoche, die wir gewohnheitsmässig mit „Alter" bezeichnen."

Nach BERENBERG-GOSSLER (1919) führt der mit Stoffwechseldefekten und Verlust der Regenerationsfähigkeit verbundene Zusammenschluss von Zellen zu einer höheren Individualität zum Tode.

Nach VORONOFF (1920) steht die Altersschwäche in engem Zusammenhang mit der Tätigkeit der Sexualdrüsen.

Nach LORRAINE (1921) ist das Alter eine Depression des ganzen Organismus.

PÜTTER (1921) ist es gelungen festzustellen, dass Alter und Sterblichkeitsprozentsatz der Individuen bei allen Arten in einem direkten Verhältnis zueinander stehen.

LACASAGNE (1921) bezeichnet als Ursache des Alterns eine Schwächung der Schutzreaktionen infolge unzulänglicher Ausscheidung der Zerfallprodukte.

„Wenn wir alle Resultate der Biologie des Todes zusammenstellen, — sagt PEARL (1922) —, so erscheint es uns als hinlänglich feststehende Tatsache, dass die Vererbung ein Faktor von erster Wichtigkeit in der Begrenzung der normalen, natürlichen Lebensdauer ist. Im Vergleiche mit diesem Faktor scheint die Wirkung der Kräfte der Umgebung weniger bemerkenswert zu sein."

Nach EHRENBERG (1923) ist der Tod das „katastrophale Ende eines Ablaufs, im grossen wie im kleinsten, und gleichgültig, ob es zugleich ein Neuanfang ist oder nicht, es ist die Unstetigkeit, der Sprung in der Geschehensfolge" und weiter: „Es ist also nicht die Anreicherung irgendwelcher mehr oder weniger nebensächlicher Schlacken, von deren Entfernung oder Ausschaltung das Schicksal des lebenden Gebildes abhängt. Es ist die Struktur selbst, die, wie sie notwendig entsteht, ja ihr Entstehen recht

eigentlich der Lebensvorgang ist, das katastrophale Ende bedingt. Je kolloidaler, um so passiver wird die Rolle sein, die die betreffende Substanz spielt, sind doch auch die Energieumsetzungen bei den Kolloidreaktionen sehr geringe. Das Problem des Todes wie das der Zellteilung ist letzthin ein chemisches Problem."

RÖSSLE (1923) sagt : „Der Tod liegt nicht an den Zellen, sondern an der Organisation, so lautet die heutige Auffassung im allgemeinen. Aber im Verlaufe der Phylogenese hat der Tod selbst durch die spezifische Organisation der höheren Vielzelligen den Charakter eines Alterstodes erhalten; vom vorhin eingenommenen Standpunkte aus bedeutet dies, dass das Individuum imstande ist, jenseits der durch Wachstum gekennzeichneten Periode seiner Entwicklung und auch lange nach Vollendung seiner der Erhaltung der Art dienenden Periode der geschlechtlichen Vollreife sein Leben zu fristen."

Nach KOPPANYI (1923) steht das Altern mit einer Störung der endokrinen Korrelation in engem Zusammenhang.

Nach KAMMERER (1921) ist die Todesursache ein und dieselbe für ein- und vielzellige Organismen : die Zerfallprodukte, die im Laufe des Stoffwechselprozesses erscheinen, häufen sich um die Zellen und innerhalb derselben an und können nicht entfernt werden. Es bleiben vorläufig noch folgende Unterschiede bestehen : die Einzelligen können unter günstigen Umständen diesem Prozess entgehen, die Vielzelligen jedoch werden ihm früher oder später unterworfen, weil er bei ihnen mit der Funktionsverteilung unter den Geweben zusammenhängt. KAMMERER unterscheidet hierbei folgende Arten von Alterstod : 1° Tod durch Abnützung, 2° Tod als Resultat von Stoffwechselprozessen, der als Folgeerscheinung chemischer Einflüsse eintritt.

DOMS (1922) bemerkt Fehlerhaftigkeit in den Versuchen, die Alterserscheinungen durch diese oder jene Altersveränderungen in der Organisation zu erklären, und nach seiner Meinung hängt die Ursache des Alterns mit jenen Momenten zusammen, die den Wiederaufbau der zerstörten Materie hemmen. Diese Momente entwickeln sich hauptsächlich im Kern, der ja der Träger der vererbten Eigenschaften ist.

Auf Grund seiner Versuche schliesst OSTERHOUT (1922) : Die Tatsachen beweisen, dass Veränderungen in der Durchdringbarkeit

eine dethermierende elektrische Leitungsfähigkeit herbeiführen können. Diese Veränderungen sind offensichtlich von Wichtigkeit, da sie alle fundamentalen Lebensprozesse angreifen können. Es ist bewiesen, dass eine Untersuchung dieser Veränderungen durch elektrische Messungen es ermöglicht, solche Begriffe wie Vitalität, Schädigung, Wiederaufbau und Tod auf quantitativem Wege zu behandeln. Es befähigt uns auch, das Verhalten der Gewebe insbesondere in Bezug auf Schädigung und Wiederaufbau voraus zu bestimmen und führt uns direkt zu einer quantitativen Theorie über den Mechanismus gewisser fundamentaler Lebensvorgänge.

Dogel (1922) ist der Ansicht, dass die Forscher nicht genügende Aufmerksamkeit verwandt haben auf die mit dem Wachstum eintretenden dauernden Veränderungen in den Zellen des sympathischen Nervensystems, obgleich gerade in den Veränderungen dieses Systems der Grund verborgen liegt, der den Organismus langsam aber sicher zum Altern und zum Tode führt.

„Wenn wir den natürlichen Tod, — sagt Robertson (1925) —, als einen durch allgemeine greisenhafte Atrophie aller Strukturen des Organismus hervorgerufenen Daseinsabschluss betrachten müssten, der sich durch allgemeine Depression und schliesslich gegenseitiges Erlöschen aller Funktionen kennzeichnet, so würde der „natürliche Tod" aller Wahrscheinlichkeit niemals eintreten. In einer Welt, wo der Organismus gegen äussere Krafteinwirkungen und invasive Tätigkeit parasitarer Organismen völlig geschützt wäre, könnten wir uns die Erscheinung des natürlichen Todes vorstellen. In der Wirklichkeit jedoch ist der Tod immer in gewissem Grade accidental."

„Mit der zunehmenden Differenzierung, — sagt Korschelt (1924) —, also mit der steigenden Entwicklungshöhe der Organismen, bieten sie eine immer grössere Angriffsfläche für innere Schädigungen, welche ihr Leben bedrohen."

Nach Gley (1924) hängt das Alter nicht, oder zum mindesten nicht völlig, vom Zustand der Geschlechtsdrüsen ab; vielmehr ist der Verfall dieser Organe eines der Alterssymptome.

Nach Jaworsky (1924) ist das Altern des Blutes die Ursache des Alters.

Biedl (1926) sagt, dass das Altwerden, der absteigende Schenkel

der Lebenskurve, ein weiterer naturgemässer Abschnitt der Entwicklungsreihe ist.

FLIESS (1924) analysiert das Problem des Todes vom Standpunkte seiner Periodenlehre und des Gesetzes der Erhaltung physischer Energie.

PLAVILSCHTIKOFF (1925) führt aus: Der Tod resultiert aus einer Stabilisation der Eiweissmoleküle. Ob diese wohl die Folge einer Veränderung der Molekularstruktur ist? Um diese Frage zu klären, müssen wir den Bau des lebendigen und toten Eiweiss kennen. Die Stabilisation der Eiweissmoleküle tritt nach LÖB als Folgeerscheinung gegenseitiger Sättigung der Aldehyden- und Aminogruppen ein.

A. PICTET benutzt und erweitert LÖBS Schlussfolgerungen und führt sie zu folgendem Endresultat: Ihrer Natur nach können die Aldehyden- und Aminogruppen nicht die Bestandteile einer geschlossenen Kette bilden. Ihre Anwesenheit im lebendigen Eiweiss erfordert das Vorhandensein einer offenen Kette. Die Verbindung beider Atomgruppen als Bestandteile ein und derselben offenen Kette kann nicht ohne Schliessung dieser Kette vor sich gehen. Stabilisierung des lebendigen Eiweisses ruft Zyklisation hervor. Wenn das Eiweiss des Zellenprotoplasma seine offenen Ketten schliesst, so gelangt es in Gleichgewichts- und Ruhezustand. Zyklisation des Eiweisses ist sein Tod.

Nach HIRSCH (1926) sind die Gesetze, nach denen sich ein früherer oder späterer Eintritt der Involution regelt, noch völlig unerforscht. Nur allgemein lässt sich sagen, dass die Konstitution, und zwar in ihren erblichen sowohl wie in ihren umweltbedingten Elementen, hierbei eine sehr wesentliche Rolle spielen mag.

PARCHON (1925) demonstriert, dass das Altern ein pathologischer Zustand ist, den wir unter die allgemeinen Distrophien neben die glandularen klassieren können, obwohl wir erkennen, dass ihr pathogenetischer Mechanismus noch complexer ist, als bei den letzten.

HARTMANN (1926) wirft zum Problem des Alters folgende Frage auf: Sind mit der Assimilation und dem Wachstum auch bei Protisten, die sich nur durch Zweiteilung vermehren, fortschreitende Entwicklungsvorgänge, also ein Altern, verbunden, und bedeutet die Fortpflanzung, bezw. die Zellteilung bereits eine Verjüngung dieser Systeme?

Nach RUZICKA (1926) ist Protoplasmahysteresis eine der Ursachen des Alterns, welche den Stoffwechsel mechanisch behindert, die Altersatrophie und das Eintreten des Todes beeinflusst.

Nach SCHMAHLHAUSEN (1926) ist das Altern eine Folge des Ausfalls eines inadäquaten Erregers, der ein Wachstum des gesamten Idioplasma hervorruft.

Nach PERTHES (1927) ist das Altern ein Entwicklungsvorgang, der ebenso von erebten Faktoren geleitet wird wie die Entstehung der so mannigfaltigen Zellen und Gewebe des Organismus aus der anfänglich scheinbar einheitlichen befruchteten Eizelle.

STEINACH (1927), sagt PERTHES, hält selbst den trotz gelungener Verjüngung schliesslich eintretenden Tod auch für einen nervösen Zelltod.

„We may say", sagt ALFRED SCOTT WARTHIN, „that senescence is due primarily to the gradually weakening energy-charge set in action by the moment of fertilization. The immortality of the germ plasm is dependent upon the renewal of this energy-charge from generation to generation."

LUMIÈRE (1928) betrachtet das biologische Alter vom Standpunkte der Kolloidchemie und drückt ihn in folgenden Thesen aus : Die Entwicklung und Flokulation der Kolloide dienen als Grundlage für die normale und pathologische Physiologie. Der Koloidalzustand bedingt das Leben, die Flokulation ist bestimmend für Krankheit und Tod.

Nach KRYLOFF (1929) ist das Altern eine Folge sowohl phylo- und ontogenetischer Einflüsse, wie auch des Zephalisationsfaktors.

Nach BELEHRADEK (1930) gibt es keinen festen Moment, von welchem der Beginn des Alters ausgeht. Die Bezeichnung „Alter" (Senilität) dient, um die letzte Etappe eines Vorganges zu bestimmen, der zugleich mit dem Beginn des Lebens anfängt und der an sich ebenso dunkel geblieben ist wie das Leben selbst. Leben = Altern ist eine Gleichung mit zwei Unbekannten, deren Lösung uns noch niemand hat demonstrieren können.

Wir haben uns bemüht, in Form einer bestimmten Arbeitstheorie alle die bisher angeführten Anschauungen und Theorien über den Prozess der Altersentstehung zu vereinen, und bereits im Jahre 1923 den Versuch gemacht, eine eigene Theorie der Genese des Alters aufzubauen, ohne bei Erforschung der Frage

in Einseitigkeit zu verfallen und unter Berücksichtigung der wichtigsten Lebensfunktionen. Unsere Erwägungen wurden auf dem 13. Internationalen Physiologenkongress in Boston (1929) vorgelegt und später in ausführlicherer Form in den Spalten der „Ergebnisseder Physiologie" (1931) veröffentlicht. In kurzen Worten sind die Grundthesen unserer Altersentstehungs-Theorie folgende :

Die Ernährungsfunktion. Die Bedingungen für den Stoffwechsel in den verschiedenen Teilen der Zelle, soweit sie sich dem Abhängigkeitsgesetz der inneren von den äusseren Speise aufnehmenden Teilen unterordnen, sind bei den einfachen und differenzierten Organismen verschieden. Bei den ersten werden die toxischen Produkte des Stoffwechsels zwischen Zelle und Medium leicht entfernt, bei den zweiten ist die Ausscheidung erschwert, deshalb ungenügend.

Die Funktion des Wechselstandes zwischen Ruhe und Tätigkeit. Dank dieser Eigenschaft der Ernährungsbedingungen fehlt bei den einfachsten Organismen die Notwendigkeit eines längeren Ruhezustandes zur Entfernung schädlicher Produkte der Lebenstätigkeit und zur Anhäufung potentieller Energie in der Bioplastik und den mit ihr verbundenen Schutzfunktionen.

Die Funktion des Lebensschutzes, ausgedrückt durch eine Reihe von Schutzreaktionen der Zelle oder ganzer Zellenkomplexe auf Erregungen, die eine Störung ihres biologischen Gleichgewichtes hervorrufen, verändern bei fortschreitender Evolution des Tierreiches ihren Charakter im Sinne einer Progression ihrer Unzulänglichkeit für den Lebensschutz. Der Wechselstand zwischen Ruhe und Tätigkeit, als eine der Funktionen des Lebensschutzes, ist bei niederen Tieren nicht hinreichend scharf ausgedrückt während bei den höheren diese Periode deutlich umgrenzt erscheint (Schlaffunktion).

Die Funktion des Wachstums und der Regeneration. Der Prozess des Schlafes, als hauptsächliche Schutzfunktion des Körpers gegen die im Wachzustand gebildeten schädlichen Erzeugnisse der eigenen Lebenstätigkeit, steht in Verbinding mit dem Prozess des Wachstums; die Wachstumskurve geht parallel mit der Kurve des nachlassenden Schlafbedürfnisses.

Die Funktion der Vererbung. Der Wachstumsprozess ist eng mit dem durch ihn bedingten Verfall der Zelle verbunden und

beide Vorgänge ordnen sich den Gesetzen der Vererbung unter. Die Vererbung wirkt bei der Evolution der genannten Funktionen verändernd auf den Mechanismus des Alterns ein, das bei den einfachsten sich auf primitive Weise ernährenden Organismen gänzlich fehlt, und folglich verändert sie auch die ursprüngliche Intensität der Schutzfunktionen (Entfernung toxischer Stoffwechselprodukte, Fehlen einer längeren Schlafnotwendigkeit und starke Neigung zur Regeneration).

Die Entstehung des Alters. Da die Schutzfunktion bei höheren Tieren eine komplizierte Form annimmt, wie z.B. bei den Erscheinungen der Immunität, Anaphylaxie und Allergie, so muss der Entstehung des Alters eine weitumfassende Erklärung gegeben werden, und zwar vom Standpunkte der im tierischen Organismus mit der Differenzierung fortschreitenden Schwächung der Immunität gegen schädliche Produkte der eigenen Lebenstätigkeit, die sich progressiv anhäufen und das biologische Gleichgewicht der Zellenkomplexe stören.

Das Problem der Altersbekämpfung besteht, von Obigem ausgehend, in der Verstärkung aller Gewebe und Organe und insbesondere solcher Gewebe, die eine feinere differenzierte Struktur haben, wie die Gewebe des Zentral-Nervensystems. Diese Gewebe haben die allerungünstigsten Ernährungsbedingungen, werden bei unzulänglichem Schlaf vor allen zuerst betroffen, halten am frühesten von allen im Wachstum inne, sind am wenigsten zur Regeneration geneigt und deshalb werden an ihnen am ehesten die Alterssymptome konstatiert.

Dr. D. Kotsovsky.

Chișinău (Rumänien)

LITERATURVERZEICHNIS

Cicero, De Senectute.
Seneca, Epist. 108.
Hufeland, Macrobiotik.
Réveillé-Paris, *zitiert nach* Broussé.
Morel, Traité des degénérescences, 1857.
Claude Bernrad, Leçons sur les phénomènes de la vie, 1865.
Martin, *Rev. de médecine*, 1881.

BÜTSCHLI, Gedanken über Leben und Tod. *Zool. Anz.*, 1882.
HAMELIN, *nach* BROUSSÉ, 1886.
DEMANGE, Étude clinique et anatomique de la vieillesse, 1889.
CAZALIS, *nach* HUCHARD, 1893.
BROUSSÉ, De l'invol. sénile, 1886.
PASCHUTIN, Handbuch der allg. u. experimental. Pathologie, 1885 (russisch).
KANSCHTAT, Die Krankheiten d. höheren Alters u. ihre Heilung, 1879.
GOETHE AL., Über den Ursprung des Todes, 1883.
MAUPAS, *Archives de Zoologie Exper.*, 1889-1890.
MALY, Über Fehler im tierischen Chemismus und über eine Ursache der beschränkten Lebensdauer. *Verein deutscher Aerzte in Prag.* Referat in *Wiener mediz. Presse*, 1888.
DURAND-FARDEL, Handbuch der Krankheiten d. Greisenalters, 1887.
BROWN-SEQUARD, Des effets produits chez l'homme par des injections sous-cutanées d'un liquide retiré des testicules frais de cobaye et de chien. *Compt. Rend. Soc. Biologie*, 1889 u. *Arch. physiol. norm. et pathol.*, Paris, 1889.
SEIDEL, Die Pathogenese, Complicationen und Therapie der Greisenkrankheiten, 1889.
KASSOWITZ, Allgemeine Biologie. Bd. I u. II, 1889.
SABATIER, Die Unsterblichkeit vom Standpunkt evolut. Naturalismus, 1897-1891.
WEISMAN, Vorträge über Descendenztheorie, 1902.
HODGE, Changes in ganglion cells from birth to senile death. Observations on Man and Honey-bee. *Journ. Physiol.*, Vol. 17, 1894-95.
BOY-TEISSIER, Leçons sur les maladies des vieillards, 1895.
DELBOEF, *nach* BOY-TESSIER.
PAWLINOFF, Die Bedeutung des Sauerstoffs für das Leben, 1896 (russisch).
MARINESCO, *Comptes Rendus de l'Académie des Sciences*, 1900.
NOTHNAGEL, Das Sterben, 1900.
BART, Cosmos, 1900.
MUHLMANN, Über die Ursachen des Alters, 1900.
KOVOSI, Über Eiweissumsatz im Greisenalter, *Zentral-Blatt f. innere Med.*, 1901.
FINOT, La philosophie de la longévité, 1901.
FRIEDMAN, Die Altersveränderungen und ihre Behandlung, 1902.
JENNINGS, Assortative mating, variability and inheritance of size in the conjugation of Paramecium. *Journ. Exp. Zool.* Vol. II, 1902.
JICKELI, Die Unvollkommenheit des Stoffwechsels als Veranlassung für Vermehrung, Wachstum, Differenzierung, Rückbildung und Tod der

Lebewesen im Kampfe ums Dasein, 1902. — Studies in the life history of Protozoa.

CALKINS, 4. Death of the species. *Journ. of Exper. Zool.*, Vol. I, 1904.

ENRIQUES, La mort, *Riv. Scienz.*, 2, 1907.

RUBNER, Das Problem der Lebensdauer und seine Beziehung zu Wachstum und Ernährung, 1904.

BUHLER, Alter u. Tod, eine Theorie der Befruchtung, *Biol. Zentralblatt.* Bd. 24, 1904.

METSCHNIKOFF, Studien über die Natur des Menschen, 1904.

SCHULTZE E., Über die Verjüngung (russisch), 1913.

R. HERTWIG, Über Parthenogenese der Infusorien und die Depressionszustände der Protozoen. *Biol. Zentralblatt.* Bd. 34, 1914.

TEICHMANN, Leben u. Tod, 1906.

POPOFF, Über die Stimulierung der Zellfunktionen, *Biol. Zentralblatt.* Bd. 2, 1922.

TRANJEN, Das Alter als abwendbare Krankheit, 1908.

HEIG, Diet und Nahrung, (russisch), 1908.

LOEB J., The Organism as a whole from the physico-chemical viewpoint, 1916.

RIBBERT, Der Tod aus Altersschwäche, 1908.

HANSEMANN D. v., Descendenz und Pathologie, 1909.

EWALD, Über Altern und Sterben, 1909.

BLOCH, Vom Tode, 1909.

ROESER, Vieillesse et longévité, 1910.

LORAND, Das Altern; seine Ursachen und seine Behandlung durch hygienische und therapeutische Massnahmen, 1911.

LORAND LOSS WOODRUFF, A summary of the results of certain physiological studies on a pedigreed race of Paramecium, *Biochem. Bull.*, I, 1911.

METALNIKOFF, Das Problem der Unsterblichkeit u. der Verjüngung in moderner Biologie, 1924 (russisch).

DASTRE, La Vie et la Mort, 1912.

HARMS, Beobachtungen über den natürlichen Tod der Tiere, *Zool. Anz.* Bd. 40, 1912.

BERGSON, L'Évolution créatrice, 1907.

CARREL, Diminuation artificielle de la concentration des protéines du plasma pendant la vieillesse, *Compt. Rend. Soc. Biol.*, 1924.

TRETIAKOFF, Unsterblichkeit und Lebensdauer, 1913 (russisch).

MINOT, The problem of age, growth and death, 1908.

SCHÄFER, Das Leben, sein Wesen, sein Ursprung und seine Erhaltung, 1913.

ERATO-SLUTZKY, Antisenetika, 1913 (russisch).

ZAVIALOFF, Tod und Unsterblichkeit, *Priroda*, 1913 (russisch).

KASCHTSCHENKO, Tod und Lebensdauer vom Standpunkt der Biologie, 1914 (russisch).

FRIEDENTHAL, Experimentelle Prüfung der bisher aufgestellten Wachstumsgesetze. *Ber. Physiol. Ges. Berlin*, 1909.

SCHEPOTIEFF, Die biochemischen Grundlagen der Descendenztheorie, 1914 (russisch).

LIPSCHÜTZ, Allgemeine Physiologie des Todes, 1915.

VERWORN, Allgemeine Physiologie, 1915.

SCHLEIP, Lebenslauf, Alter und Tod des Individuums. Kultur der Gegenwart. *Allg. Biol.* 3, IV, Bd. I, 1915.

CHILD, Senescence and rejuvenescence based on experiments with planaria. *Arch. f. Entwicklungsmech.*, Bd. 31, 1911.

VORONOFF, Vivre, 1920.

LAURENT, La science de la vie et la longévité, 1921.

PÜTTER, *nach* KAMMERER, 1923.

LACASSAGNE, La verte vieillesse, 1921.

PEARL, The biology of death, 1922.

EHRENBERG, Theoretische Biologie, 1923.

ROESSLE, Wachstum und Alter, 1923.

KOPPANYI, Theoretische Erwägungen über die Entstehung der Alterserscheinungen und des Todes. *Naturwissenschaftl. Wochenschrift*, 21, Bd., 1922.

KAMMERER, Der Tod u. Unsterblichkeit, 1923.

DOMS, Über Altern, Tod und Verjüngung, *Ergeb. Anat. u. Entwicklungsgeschichte*, Bd. 22, 1921.

OSTERHOUT, Injury, Recovery, and Death in Relation to Conductivity and Permeability, 1922.

DOGEL, Alter und Tod, (russisch) 1922.

ROBERTSON, The chemical basis of growth and senescence, 1923.

KORSCHELT, Lebensdauer, Altern und Tod, 1924.

GLEY, Le problème de la senescence et la doctrine des sécrétions internes. *Clinique et Laboratoire*, No. 7, 1924.

JAWORSKI, La regénération de l'organisme humain par les injections de sang, 1925.

BIEDL, Jungsein und Altwerden. NI. Urania Bücherei, Prag, 1926.

FLIESS, Vom Leben und vom Tod, *Biologische Vorträge*, 1924.

PLAVILSCHTIKOFF, Tod und Unsterblichkeit, 1925 (russisch).

HIRSCH, Das Altern und Sterben des Menschen (Handbuch der normalen und pathologischen Physiologie. Bd. 17, Correlationen III), 1926.

PARCHON, Batraneta, Senilitatea, Psychozele varstei de involutie, Dementa senila, Dementa arterio-sclerotica, 1926.

HARTMANN, Über experimentelle Unsterblichkeit von Protozoen-Individuen. *Die Naturwissenschaft*, Heft 19, 1926.
RUZICKA, Starnuti a omlazeni, 1926.
SCHMALHAUSEN, Tod und Unsterblichkeit (russisch), 1925.
PERTHES, Über den Tod, 1927.
STEINACH, *zitiert nach* PERTHES.
LUMIÈRE A., La vie, la maladie et la mort, 1928.
KRYLOW, Étude actuelle sur la vieillesse humaine, (bulgarisch), 1929.
BELEHRADEK, Le mécanisme biologique du vieillissement et de la mort, *Médecine pratique*, juin, 1930.
WARTHIN, ALFRED SCOTT, The Pathology of the *Aging Process*. The Wesley M. Carpenter Lecture, 1928.
KOTSOVSKY, Die Genese des Alters, 1923 (russisch).
KOTSOVSKY, The origin of senility, *Amer. Jour. of Physiolog.*, Oct., 1929.
KOTSOVSKY, Allgemeine vergleichende Bilogie des Alters. (Genese des Alters) *Ergeb. d. Physiologie*, Bd. 31, 1931.
KOTSOVSKY, Gehirn und Alter. Zeitschr. f. d. gesam. Neurologie u. Psychiatrie. 133 Bd. 5.H. 1931.

Hall, G. S.

Notes on the History of the Concept of Thermal Conductivity

ABSTRACT

The concept of thermal conductivity required nearly three hundred years for its development and involved the expenditure of an immense amount of mental energy and experimental effort. GALILEO, HOOKE, NEWTON, BOERHAAVE, FRANKLIN, BLACK, RUMFORD, FOURIER—giants in the scientific world—contributed to the development of this concept.

During the process of refinement and abstraction of ideas concerning the transfer of heat the thermometer and calorimeter were invented and the technique of their use perfected. The preliminary concepts of heat and flow of heat, of temperature as distinct from heat, of latent and specific heats, had to emerge. Conduction, as a method of heat transference, had to be separated and distinctly set apart from radiation and convection.

Outmoded conceptions of heat served as guides in framing experiments and evaluating ideas. Theories rose and fell, arguments frequently waxed acrimonious, notions were confused; yet from this our present concept emerged, simple, useful, independent of any theory of heat.

The concept of thermal conductivity should not be viewed as the sudden, almost spontaneous, crystallization of the complete idea from a magma of confused and clouded thought. It did not, like Minerva, spring full grown from the mind of some Jove of science; more like Topsy, it just grew. It appeared simply as a late step in the slow process of refinement and abstraction of ideas concerning the transfer of heat from one body to another.

To the present student or investigator, attempting to enter fully into the history of the concept, difficulties are presented by his own familiarity with expressions, terms, prerequisite concepts and operations, now a part of his general equipment, which came into existence during the evolution of the concept. Additional difficulties are presented by his familiarity with those facts, the discovery of which resulted in the formulation of the

concept, and by his familiarity with facts almost innumerable, previously unknown, which have been revealed as a consequence of the concept.

Before the concept of thermal conductivity could emerge certain preliminary concepts and operations had to be developed. Among these we find the concept of heat itself, the concepts of temperature, of specific and latent heats, and of the flow of heat. In addition, instruments and operations had to be developed for measuring temperatures and quantities of heat. That is, the preliminary concepts had to be put on a quantitative basis.

Undoubtedly, from early times people were cognizant of the flow of heat. However, no attempt to study this process from a quantitative standpoint, or to separate the more inclusive phenomenon into the presently accepted component phenomena, could be made until the thermometer was invented.

The development of the thermometer (1), and the accompanying scale, aroused the interest and taxed the ingenuity of the greatest scientists over a period of three hundred years. A primitive form of the thermometer was invented by GALILEO about 1597 and development was gradual, although not uniform, until the establishment of the thermodynamic scale late in the nineteenth century.

Following the invention of the first thermometer, or rather thermoscope, investigators were in position to attack the problems of heat flow, although it was not until one hundred and seventy five years later that the distinction between heat and temperature was made, and during this period no attempt was made to measure heat independently of temperature.

HOOKE (2), by the aid of crude thermometers, established in 1664 the constancy of the freezing point, and in 1684 the constancy of the boiling point, of water. The first step in the resolution of the flow of heat was the attempt of MARIOTTE to examine radiation (3). In 1682 he mentioned, at a meeting of the French Academy of Sciences, that the heat of a fire reflected by a burning mirror is sensible in its focus; but when a pane of glass is inter-

(1) BOLTON, H. C., "Evolution of the Thermometer" (1900).

(2) BLACK'S Chemistry, edited by JOHN ROBISON. 1st Am. ed. (1807), p. 52.

(3) THOMSON, T., "An Outline of the Science of Heat and Electricity" 1830, p. 130.

posed between the mirror and its focus the heat is no longer sensible. The phenomenon of thermal radiation had long been known but this was the first attempt at experimental investigation.

NEWTON realized the importance of improving the thermometer scale and published his " Scala graduum caloris et frigoris " in the *Philosophical Transactions* for 1701. From his experiments he inferred that the excess temperatures of iron, heated and exposed to the air, form a geometrical progression at times which are in arithmetical progression. AMONTONS continued and extended the work of NEWTON.

FAHRENHEIT began his work on thermometers and scales in 1706, adopting the scale now known by his name in 1724. At least part of his researches in heat were carried on in conjunction with BOERHAAVE. In his " Elementa Chemiae," (4) published in 1732, BOERHAAVE discusses an experiment carried out at his suggestion by FAHRENHEIT. From the data obtained he drew erroneous conclusions as to the heat capacities of various bodies. Correct inferences were not drawn until thirty years later, and then by BLACK. Dr. MARTINE of St. Andrews published his " Essay on the Heating and Cooling of Bodies " in 1739. His experiments were similar to those of NEWTON and FAHRENHEIT and, like them, were reinterpreted by BLACK. His experiments were an improvement, both in conception and in execution, over the former ones.

In 1721, SWEDENBORG published three small Latin treatises dealing with questions in physics and chemistry (5). In the second, entitled " New Observations and Discoveries Respecting Iron and Fire, etc.," he gives a good description of the iron furnace and its operation, and attempts a mathematical analysis of the progress of heat throughout the cross-section of the furnace. He presents a theory of heat and several laws, referring almost wholly to convection. As in the case of other experimenters of the period, he did not appreciate that he was dealing with convection and temperature flow.

An historical study indicates that the Heat-Electricity analogy

(4) BOERHAAVE, H., " Elementa Chemiae," 2nd ed. (CAVELIER) (1733), p. 75 ff. BLACK refers particularly to Exp. 20, Coroll. 11, p. 144.

(5) SWEDENBORG, E., " Principles of Chemistry, etc." Trans. by C. E. STRUTT (1847).

has consistently proved fruitful in both fields. The beginning of experimental study in electricity was late compared to that in heat. STEPHEN GRAY laid the true foundation of electricity as a science in 1720. His discoveries were published in the *Philosophical Transactions* from 1720 to 1736. In 1729, he announced his discovery of electrical conduction, introducing the terms " electrics per se " and " non-electrics." (6) The terms " conductor " and " non-conductor " were invented and introduced by DESAGULIERS about 1737-40 (7). They were soon adopted in the field of heat.

Even though the flow of electricity had been established, experimenters were handicapped, as were those in heat before the time of GALILEO, by the lack of measuring instruments. GRALATH, and independently RICHMANN, succeeded in developing crude instruments about 1745, although the first practical measuring instrument was given to electrical science by HENLEY about 1770. It is described by PRIESTLEY in a letter to FRANKLIN (8).

As revealed through his letters, FRANKLIN early formed, as a result of experiment, rather accurate ideas concerning electrical conduction. Also, at an early date, he became interested in the phenomena of heat, although he did not pursue this subject as far as he did electricity.

FRANKLIN was apparently the first to invoke the Heat-Electricity analogy. This was done in connection with conduction. As early as 1751 FRANKLIN had extended the list of electrical conductors and by 1755 had demonstrated that some substances were better conductors than others. This put Heat and Electricity on a par and from then on the race to the conception of conductivity was nip and tuck. In a letter to JOHN LINING, written in 1757, he brings forward the analogy (9) : " Professor SIMSON, of Glasgow, lately communicated to me some curious experiments of a physician of his acquaintance, by which it appeared, that an extraordinary degree of cold, even to freezing, might be produced by evaporation,—".

---

(6) GRAY, S., *Trans. Roy. Soc.* London 37 (18) 1732.

(7) PRIESTLEY, J., " History and Present State of Electricity, etc." 5th ed., (1794), p. 60.

(8) PRIESTLEY, J., *Trans. Roy. Soc.* London 62 (359) 1772.

(9) FRANKLIN, B., The works of, ed. by SPARKS, vol. 6, (1840), p. 204.

"I know not how this phenomenon is to be accounted for, but it gives me occasion to mention some loose notions relating to heat and cold, which I have for some time entertained, but not yet reduced into any form. Allowing fire, as well as electrical, to be a fluid capable of permeating other bodies, and seeking an equilibrium, I imagine some bodies are better fitted by nature to be conductors of that fluid than others; and that, generally, those which are the best conductors of the electrical fluid, are also the best conductors of this; and e contra." He gives several examples of poor and good thermal conductors and details an experiment which shows that lead is a better conductor than wood.

FRANKLIN was also, apparently, the first to experimentally establish the connection between heat and electricity (10). It is true that the actual experiment was carried out by KINNERSLEY, in 1760; but at the suggestion of, and as designed by, FRANKLIN. FRANKLIN frequently suggested experiments in this manner and as a result a number of experiments, now classical, conceived and interpreted by FRANKLIN, are known by the names of those who performed them.

FRANKLIN's first experiments on the relative conducting powers of different materials for heat and electricity were merely qualitative. He immediately proceeded to place them on a quantitative basis. PRIESTLEY, in the fifth edition of his "History and Present State of Electricity" (1794) says (11): "In a conversation I once had with Dr. FRANKLIN, Mr. CANTON, and Dr. PRICE, I remember asking whether it was probable that there was any difference in the conducting power of different metals; and if there was, whether it was possible to ascertain that difference? I have since endeavoured to carry into execution a scheme proposed by Dr. FRANKLIN." Following this he details the experiments and results, with criticisms and suggestions for improvement. The first edition of the "History" (1767) makes no mention of this suggestion of FRANKLIN, so the exact date is somewhat in doubt.

(10) FRANKLIN, B., *loc, cit.*, vol. 5, (1837), p. 373.

(11) PRIESTLEY, J., "History and Present State of Electricity, etc." 5th ed. (1794), p. 636.

Dr. INGENHOUSZ published his " Nouvelles Expériences " in 1780. This included a report on his experimental determination of the relative thermal conductivities of various metals. The experiment was suggested to INGENHOUSZ by FRANKLIN and was similar in conception to the one outlined by PRIESTLEY for electricity (12).

The value of FRANKLIN's thermal experiments was lessened, as in the case of all others, by the current confusion between temperature and heat. In addition, at least previous to FRANKLIN, the concept of heat capacity, or specific heat, had not been formulated and there was no good method of measuring quantities of heat. The confusion between temperature and heat was first clearly perceived by BLACK about 1760. It is rather surprising that FRANKLIN was unfamiliar with the views of BLACK. They were contemporaries and FRANKLIN corresponded, in regard to heat phenomena, with SIMSON, also of Glasgow, and with CULLEN, who was BLACK's sponsor and predecessor at both Glasgow and Edinburgh. Both men were on friendly terms with LAVOISIER. It is true BLACK did not publish his scientific findings, being, as his biographer ROBISON says (13) " perhaps fastidiously nice in his notion of a philosophical performance, and too severe in his observations on the hurried publications of some chemists, which he used to call slovenly, and to consider as literary manufacture for profit."

BLACK made remarkable contributions to the field of heat during the ten years following 1756. By means of clear thinking and a very few " critical " experiments he resolved the confusion of heat and temperature, conceived of thermal equilibrium, advanced his doctrine of specific heats, developed his method of mixtures and established the concept of latent heats. In commenting on the " distribution of heat " BLACK says (14): " This is what has been commonly called an equal heat, or the equality of heat among different bodies; I call it the equilibrium of heat. The nature of this equilibrium was not well understood, until I pointed out a method of investigating it. Dr. BOERHAAVE

(12) FRANKLIN, B., *loc. cit.*, vol. 6, p. 438.
(13) BLACK, *loc. cit.*, preface, p. l.
(14) *Ibid.*, pp. 74, 75.

imagined, that when it obtains, there is an equal quantity of heat in every measure of space, however filled up with different bodies;—." " But this is taking a very hasty view of the subject. It is confounding the quantity of heat in different bodies with its general strength or intensity, though it is plain that these are two different things, and should always be distinguished, when we are thinking of the distribution of heat."

Concerning heat capacities, he states (15) : " It was formerly a common supposition, that the quantities of heat required to increase the heat of different bodies by the same number of degrees, were directly in proportion to the quantities of matter in each; and therefore, when the bodies were of equal size, the quantities of heat were in proportion to their densities. But very soon after I began to think on this subject, (anno 1760), I perceived that this opinion was a mistake,—." " The inference which Dr. BOERHAAVE drew from this experiment is very surprising. As soon as I understood this experiment in the manner I have now explained it, I found a remarkable agreement between it and some experiments made by Dr. MARTINE (Essay on the Heating and Cooling of Bodies) which appeared at first very surprising and unaccountable,—". BLACK's conclusions were always rigorously but sparingly supported by experimental evidence. In the words of ROBISON (16), " Dr. BLACK always expressed a dislike to the bringing forward of a multitude of experiments of the same kind, saying that he felt his own confidence more forcibly won by one judicious and simple experiment that by any number of examples of inferior evidence."

BLACK shows a clear conception of temperature, quantity of heat and specific heat throughout his discussion, although he clothes his ideas in the old phraseology. He frequently speaks of the " communication " or " transmission " of heat, but these notions cover more than our present " conduction." He realized that all substances did not " transmit " heat with the same " celerity." In fact, he says (17) : " Fluids generally receive heat, and transmit it through them more quickly, than the greater number

(15) *Ibid.*, pp. 76, 78.
(16) *Ibid.*, preface, p. xxvi.
(17) *Ibid.*, p. 87.

of solids." This indicates the then current confusion between conduction and convection.

The remarkable progress made by BLACK was due, in great part, to his invention and use of the calorimeter (1760). As in the case of the first thermometer, the instrument was improved by others, particularly LAVOISIER and LA PLACE, and the technique of its use developed. Again, as with the thermometer, considerable time elapsed before the establishment of a suitable scale or unit for heat quantity. The scientific world was now for the first time in position to measure the thermal conduction of bodies.

Scientists had early isolated radiation as an independent method of transferring heat. After a lapse of eighty-five years, LAMBERT repeated and greatly extended the experiments of MARIOTTE (3). SCHEELE, in his " Treatise on Air and Fire," published in 1777, made many important and new observations on the radiation of heat. It took much longer to separate the phenomena of conduction and convection.

RUMFORD, in his enthusiastic study of thermal phenomena, determined the conducting power of a large number of substances. His choice of materials was most catholic, ranging from metallic and non-metallic solids through liquids and gases to clothing, stews, mushes and apple sauce, the latter in various " concentrations." The origin of his interest in the problem may be indicated by the following extract from his " Essays " (18) : " When dining, I had often observed that some particular dishes retained their Heat much longer than others; and that apple-pies, and apples and almonds mixed,—(a dish in great repute in England)—remained hot a surprising length of time. Much struck with this extraordinary quality of retaining Heat, which apples appear to possess, it frequently recurred to my recollection; and I never burnt my mouth with them, or saw others meet with the same misfortune, without endeavouring, but in vain, to find out some way of accounting, in a satisfactory manner, for this surprising matter."

To RUMFORD we owe the distinction between conduction and convection, and he describes vividly in his " Essays " the experi-

(18) RUMFORD, " Essays, Political, Economical, and Philosophical," 1st Am. ed. (1799), vol. 2, p. 202.

ments which led him to the concept of convection. In his early papers he assumed that all heat, not radiated, was transferred by conduction. In his " New Experiments on Heat " he says (19) : " Examining the conducting power of air, and of various other fluid and solid bodies, with regard to heat,—". " It having been my intention from the beginning to examine the conducting powers of the artificial airs or gases,—". " Does humidity augment the conducting power of air? " " From these experiments it appears, that the conducting power of air is very much increased by humidity." " This is certainly a subject of investigation not less curious in itself than it is interesting to mankind; and I wish that what I have done may induce others to turn their attention to this long neglected field of experimental inquiry. For my own part, I am determined not to quit it."

As a result of subsequent experiments, ingeniously designed to determine the point and well executed, he concluded that liquids and gases did not conduct heat. In his own words (18), " —although the particles of any fluid, individually, can receive heat from other bodies, or communicate it to them; yet, among these particles themselves, all interchange and communication of Heat is absolutely impossible." Also, " When heat is propagated in solid bodies, it passes from particle to particle, de proche en proche, and apparently with the same celerity in every direction; but it is certain that heat is not transmitted in the same manner in fluids (20) " Of these experiments ROBISON says (21) : " We must not omit taking notice of a very singular distinction, which Count RUMFORD has endeavoured to establish between the conducting power of solid and fluid bodies, namely, that while heat is propagated along solid bodies, from particle to particle, according to certain regular laws, fluid bodies are heated in their remote parts only by the actual transference of the particle which received heat from the hot body while in contact;—. The ingenious and worthy author has supported his opinion by a series of curious and well contrived experiments, the results of which are most unexpected and surprising. Dr. BLACK was much struck with the novelty and importance

(19) RUMFORD, *Trans. Roy. Soc.*, London 76 (273-304) 1786.
(20) RUMFORD, *Nicholson's J.* 14 (353-363) 1806.
(21) BLACK, *loc. cit.*, p. 339.

of this opinion, and began to examine it with that cautious exactness which he had always practiced; but the great decline in his health—." " It is clearly established by the experiments of Count RUMFORD, that when the internal motion, producible by the inequality of specific gravity, occasioned by heating the lower parts of a fluid, is prevented, the progress of heating is almost insensible, and is no more than we may expect by means of the solid matter of the vessel and by radiation." " There is nothing, therefore, in the opinion offered with great caution by Count RUMFORD, that is incompatible or inconsistent with our most distinct notions on these subjects. It is a fair field of inquiry, most ingeniously laid open to philosophers." This conclusion of RUMFORD's was based upon the best experimental evidence available and persisted in part until 1880. It deserves extended mention because it indicates that " the next decimal " will not only improve precision but may markedly modify ideas.

In the planning of his experiments RUMFORD made use of the Heat-Electricity analogy, calling attention to it on several occasions (19). He kept in close touch with his friend PICTET, who first proved that the conducting power of a body was a property, independent of physical position (22). By this time, approximately 1810, the concept of thermal conduction had been fully clarified, principally due to the work of RUMFORD and PICTET. There now remained the final step, making the property specific and the conception of conductivity.

" In the fullness of time " (1822) FOURIER produced his " Theorie Analytique de la Chaleur," in which he " treated the subject of the propagation of heat with a power and completeness which left little room for extension or improvement and suggested or rendered possible all the later developments (23)." FOURIER was the first to give a thoroughly scientific definition of conducting power. For a period of fifteen years, culminating in his " magnum opus," he worked at experimentation and interpretation. " The work which we now publish has been written a long time since;—" (24) " I have deduced these laws from a prolonged

(22) RUMFORD, " Essays,..." vol. 2, p. 241.
(23) PRESTON, " Theory of Heat " 3rd ed., (1919), p. 605.
(24) FOURIER, " Analytical Theory of Heat," Trans. FREEMAN (1878), p. 9.

study and attentive comparison of the facts known up to this time : all these facts I have observed afresh in the course of several years with the most exact instruments that have hitherto been used." (25)

" The mathematical theory of heat includes : first, the exact definition of all the elements of the analysis; next, the differential equations; lastly, the integrals appropriate to the fundamental problem." (26) " The numerical interpretation of the results of analysis is, however, necessary, and it is a degree of perfection which it would be very important to give to every application of analysis to the natural sciences. So long as it is not obtained, the solutions may be said to remain incomplete and useless, and the truth which it is proposed to discover is no less hidden in the formulae of analysis than it was in the physical problem itself." (27) " To found the theory, it was in the first place necessary to distinguish and define with precision the elementary properties which determine the action of heat." (25) " What are the elementary properties which it is requisite to observe in each substance, and what are the experiments most suitable to determine them exactly ? " (28) Speaking of his coefficients, he says, " These three elementary qualities are represented in our formulae by constant numbers, and the theory itself indicates experiments suitable for measuring their value. As soon as they are determined, all the problems relating to the propagation of heat depend only on numerical analysis. The knowledge of these specific properties may be directly useful in several applications of the physical sciences; it is besides an element in the study and description of different substances. It is a very imperfect knowledge of bodies which ignores the relations which they have with one of the chief agents of nature." (29)

FOURIER had very pronounced, but justifiable, ideas as to the permanent value of his work. He was fully aware of the experimental limitations set upon his work and predicted extensions which have since been made. " Observations more exact and more

(25) *Ibid.*, p. 2.
(26) *Ibid.*, p. 454.
(27) *Ibid.*, p. 21.
(28) *Ibid.*, p. 4.
(29) *Ibid.*, p. 19.

varied will presently ascertain whether the effects of heat are modified by causes which have not yet been perceived, and the theory will acquire fresh perfection by the continued comparison of its results with the results of experiment; it will explain some important phenomena which we have not yet been able to submit to calculation;—." (30) " It cannot be doubted that the coefficient *k* (FOURIER's coefficient of conductivity), the measure of the permeability, is subject to sensible variations; but on this important subject no series of experiments has yet been made suitable for informing us how the facility of conduction of heat changes with the temperature and with the pressure." (31)

FOURIER limited his experimental and theoretical work almost entirely to solids, although he admits that liquids have the property of conductivity in a small degree, but masked. He does not mention gases at all. He recognizes heat transfer by convection but does not use the term.

OHM, in 1828, laboring independently, although probably cognizant of the work of FOURIER, produced his paper dealing with electrical conductivity, thus keeping Electricity abreast of Heat (32).

While the first determinations of the thermal conductivity of solids were probably made by FOURIER, the first determinations in which precision was stressed were those of FORBES (33). He invoked the Heat-Electricity analogy in establishing the correlation of thermal and electrical conductivities for solids (34); in addition he was the first to show the variation of thermal conductivity with temperature (35).

It required a great deal of experimental data to establish the fact, in opposition to the early opinion of RUMFORD, that liquids and gases would conduct heat. Both MURRAY (36) and RUMFORD (20) showed that heat would pass downwards through a

---

(30) *Ibid.*, p. 25.

(31) *Ibid.*, p. 465.

(32) OHM, G. S., *Schweigger's, J.* 46 (137-166) 1826.

(33) FORBES, J. D., *Trans. Roy. Soc. Edin.* 16 (189-236) 1849.

(34) FORBES, J. D., *Phil. Mag.* 4 (15-28) 1834, Esp. p. 27.

(35) FORBES, J. D., *Brit. Assoc. Adv. Sci.* 22 (260-261) 1853; *Trans. Roy. Soc. Edin.* 23 (133-146) 1864.

(36) MURRAY, J., *Nicholson's J.* 1 (165-173) 1802, 1 (241-251) 1802.

liquid. The experiments of THOMSON (37) and TRAILL (38) also indicated this, although the interpretation of the data was still questioned. DEWEY was the first American scientist to enter the field (39). He criticised the experiments of MURRAY but supported his conclusions with new data. DESPRETZ seems finally to have established the fact (40).

RUMFORD's conclusion in the case of gases was harder to overthrow. ANDREWS (41) and GROVE (42) were the first to obtain data bearing on this point. However, the first experiments deliberately designed to answer this question were those of MAGNUS (43), using hydrogen as the conducting medium. His interpretation of his results was that gases conduct heat in the same way as do metals. TYNDALL discusses in detail the experiments of MAGNUS and concludes : (44) " Beautiful and ingenious as these experiments are, I do no think they conclusively establish the conductivity of hydrogen." In the sixth edition of his book, with the preface dated by TYNDALL in 1880, there is no change in his position on the work of MAGNUS, even though in the meantime it had been quite well established, both theoretically and experimentally, and with good agreement, that gases conduct heat !

In conclusion it is worth noting that the concept of thermal conductivity is entirely independent of any theory of heat. It was developed, for the most part, by men holding antiquated and, as we view them, erroneous, notions as to the nature of heat. Yet, strange as it may seem, it was only out of these notions that the concept grew.

Massachusetts Institute of Technology, ALEX. C. BURR.
Cambridge, Mass.

---

(37) THOMSON, T., *Nicholson's J.* 4 (529-545) 1801.

(38) TRAILL, T. S., *Nicholson's J.* 12 (132-139) 1805.

(39) DEWEY, C., *Am. J. Sci. & Arts* 28 (151-155) 1835; *Sturgeon's Ann. of Elec.* 2 (126-127) 1838.

(40) DESPRETZ, C., *Ann. de Chimie et de Physique* 71 (206-222) 1839.

(41) ANDREWS, T., *Proc. Roy. Irish Acad.* 1 (465-468) 1841.

(42) GROVE, W. R., *Phil. Mag.* 27 (442-446) 1845; 35 (114-126) 1849.

(43) MAGNUS, G., *Phil. Mag.* 22 (1-21) 1861; 22 (85-106) 1861.

(44) TYNDALL, J., " Heat Considered as a Mode of Motion." 1st ed. (1863), p. 240.

INDEX OF NAMES

Amontons, G. 248,
Andrews, T. 258,
Black, J. 246, 248, 251, 252, 253, 254,
Boerhaave, H. 246, 248, 251, 252,
Canton, J. 250,
Cullen, W. 251,
Desaguliers, J. T. 249,
Despretz, C. 258,
Dewey, C. 258,
Fahrenheit, G. D. 248,
Forbes, J. D. 257,
Fourier, J. B. J. 246, 255, 256, 257,
Franklin, B. 246, 249, 250, 251,
Galilei, G. 246, 247, 249,
Gralath, D. 249,
Gray, S. 249,
Grove, W. R. 258,
Henley, W. 249,
Hooke, R. 246, 247,
Ingenhousz, J. 251,
Kinnersley, E. 250,
Lambert, J. H. 253,
LaPlace, P. S. 253,
Lavoisier, A. L. 251, 253,
Lining, J. 249,
Magnus, G. 258,
Mariotte, E. 247, 253,
Martine, G. 248, 252,
Murray, J. 257,
Newton, I. 246, 248,
Ohm, G. S. 257,
Pictet, M. A. 255,
Price, R. 250,
Priestley, J. 249, 250, 251,
Richmann, G. W. 249,
Robison, J. 251, 252, 254,
Rumford, 246, 253, 255, 257, 258,
Scheele, K. W. 253,
Simson, R. 249, 251,
Swedenborg, E. 248,
Thomson, T. 258,
Traill, T. S. 258,
Tyndall, J. 258.

Notes and Correspondence

I. — QUERIES

In an attempt to revive and stimulate this department of *Isis*, the Editor sent, on December 19, 1932, to some of thec ontributors a letter containing the following statements :

Scholars interested in the History of Science constitute a highly heterogeneous group, and the editor of a journal devoted to their very complex and abundant needs cannot hope to satisfy them all except by insuring as much variety in its contents as possible. Of course, the Critical Bibliography ought to satisfy almost every need, yet we need something more.

I would like to be able to publish a larger number of short articles or notes, and I am especially anxious to increase the number of queries. Though the number thus far published is very small, their usefulness has been amply confirmed. For example, a question concerning GIOVANNI FONTANA, which appeared in vol. 7, p. 105, provoked the publication of two important papers by LYNN THORNDIKE (vol. 15) and A. BIRKENMAJER (vol. 17), and a third paper on the same subject is now being printed. Indeed nothing is more stimulating than a good query.

In order to emphasize their value I am trying to collect a large number of them, and would be very much obliged to you if you would kindly send me a few pertinent ones. Single queries risk being overlooked, but if I could publish, say, a dozen of them in each number, the readers would look forward to them and would be more inclined to send answers or additional ones. In this simple way, the vitality and attractiveness of our journal would be much increased. Moreover the publication of a query is often the best way of announcing that an investigation is undertaken and of preventing duplication.

Queries and answers should be neatly typed on one side of the paper only, a separate sheet being devoted to each item.

The Editor has prepared the following questions which may serve as illustrations.

Query no. 23.—*Arabic "commercial" arithmetic.*

In my *Introduction* (vol. 1, 668, 715), I mentioned two treatises on mu'āmalāt, and in both cases I translated that word by "commercial arithmetic." Some time ago as I was reading the Kitāb ṭabaqāt al-umam of the qāḍī ABŪ-L-QĀSIM ṢĀ'ID IBN AḤMAD AL-ANDALUSĪ (see *Intr.* 1,

776), I came thrice across a Kitāb al-mu'āmalāt, and wondered whether my translation was correct (see p. 68, 69, 70 of CHEIKHO's edition, Beirūt 1912).

In FREYTAG's Lexicon mu'āmalāt is translated " pecunia signata quae in mercatura admittitur "; in the Dictionnaire of A. DE BIBERSTEIN KAZIMIRSKI, mu'āmalāt=effets de commerce; LANE is silent on the subject; in DOZY's Supplement, the word is translated affaires de commerce, argent, negotia iuridica, etc.

I know altogether of four treatises on mu'āmalāt (and there may be more) as follows :

1. By ABŪ-L-QĀSIM MASLAMA IBN AḤMAD AL-MAJRĪṬĪ (d.c. 1007). See my *Introduction*, 1, 668 ; Ṭabaqāt, p. 69, wherein it is said that he wrote a book " on the whole of the science of numbers which is called among us al-mu'āmalāt " (it is this much broader definition which aroused my suspicions).

2. By ABŪ-ḤASAN 'ALĪ IBN SULAIMAN AL-ZAHRĀWĪ, a pupil of the former. See Ṭabaqāt, p. 70, where his book is called " Kitāb sharīf fi-l-mu'āmalāt 'ala ṭarīq al-burhān " (the noble book on mu'āmalāt in the demonstrative manner.) According to IBN ABĪ UṢAIBI'A (MÜLLER's edition, vol. 2, 40) that book was also called Kitāb al-arkān (book of foundations). See also SUTER no. 190.

3. By ABŪ-L-QĀSIM AḤMAD IBN MUḤAMMAD called AL-ṬUNBARĪ (died c. 1025). See Ṭabaqāt, p. 68 ; the consonantal spelling Ṭunbarī being followed by the interpolation kadhā (sic). See SUTER no. 188, where he is called IBN AL-ṬONEIZĪ (?).

4. By ABŪ-L-QĀSIM AṢBAGH IBN MUḤAMMAD IBN AL-SAMḤ (died in 1035). See *Introd.*, 1, 715 ; Ṭabaqāt, p. 69, where his book is called the fruit of numbers or al-mu'āmalāt.

It is very remarkable that these four treatises (the only ones I know of) are all of them productions of the same country, Spain, and of the same time, the first half of the eleventh century (or perhaps the end of the tenth century).

To solve the meaning of the word mu'āmalāt, it would be necessary to examine these treatises, but unfortunately none of the MSS. is available to me. I much hope that one of them may be some day analyzed and discussed in *Isis*.

Finally I should like to broaden the question. What was the nature of the " commercial arithmetic " of the Arabs? Considering that our main sources on the subject seem to be Spanish, one of our Spanish contributors might perhaps be induced to investigate it. This might involve interesting developments concerning mediaeval commerce, and the transmission of Muslim commercial methods to the Christian world.

The reality of such a transmission is already substantiated by many Spanish commercial terms of Arabic origin.

There might also be some points of contact between the treatises on mu'āmalāt and those on ḥisba of which a Spanish exemplar of the same (eleventh) century has been recently edited by G. S. COLIN and E. LÉVI-PROVENÇAL (Paris, 1931 ; *Isis*, 19, 528).

GEORGE SARTON.

Query no. 24.—*Hippocratic oath in Arabic.*

In his charming little volume on the Doctor's Oath, W. H. S. JONES (1) quotes the Arabic version of it given by IBN ABĪ UṢAIBI'A in his 'Uyūn al-anbā' (2), a book composed about the middle of the thirteenth century. But considering the long and deep interest of the Arabic writers in HIPPOCRATES there must be an earlier text than that. It would be worthwhile to retrace the Arabic and Jewish traditions on the subject. It should be noted that the Doctor's Oath is a part of the Hippocratic tradition, but not of the Galenic one.

More generally the history of Hippocratic traditions in the Arabic writings remains to be written. Readers of my *Introduction* will realize how large and fascinating a subject this would be ; they will obtain another glimpse of it in Baron CARRA DE VAUX's article Buqrāṭ in the Encyclopaedia of Islām (vol. 1, 784, 1912).

GEORGE SARTON.

Query no. 25.—*Orientation of the miḥrāb in mosques.*

When the Prophet instituted the ṣalāt (prayers), he ordered them to be said in the direction of Jerusalem. However, in Rajab or Sha'bān of the year 2,when he finally despaired of Jewish coöperation and decided to turn his back on them, he changed the qibla (the direction of prayer) from Jerusalem to Mecca. From that time on every Muslim is enjoined to say his prayers in the direction of Mecca. If he be in a mosque, that direction is supposed to be given to him by some mark or more generally by the miḥrāb ; if he be outside, it is his duty to pray in that direction as nearly as he is able to determine it.

At the beginning, the qibla was simply indicated by a slab of stone or a strip of paint on the wall. So it was in the first mosque of Medīna, where the qibla was first indicated on the northern wall (toward Jerusalem), then on the southern wall (toward Mecca). Gradually the qibla became a miḥrāb. In the eighth century, it already had its usual form :

(1) Cambridge 1924; see *Isis*, 11, 154; *Introduction*, 2, 95.
(2) *Introduction*, 2, 685.

a semi-circular shell-shaped niche flanked by pillars and often richly ornamented. It is possible that the miḥrāb is the development (or rather the atrophy) of the apse of a basilica ; though its form is so natural that it is hardly necessary to postulate any kind of imitation.

Considering the religious duty distinctly stated in the Qur'ān (2, 136, etc.) one would expect the miḥrāb to be very accurately oriented in every mosque. That was my own assumption and I even used it as an argument to account for the early development of astronomical studies in the Dār al-Islām. Indeed the determination of the qibla implies a fair amount of mathematical and astronomical knowledge (1). Therefore I was amazed when I discovered in the course of my travels in the East that the miḥrāb direction was often inaccurate, and even grossly so.

In Syria this might be partly explained by the fact that many Christian churches were turned into mosques. This may account for the transmission of the apse form to the miḥrāb, but not for the position of the latter. In reality in mosques derived from churches the qibla is perhaps more exactly determined than in others, this being a part of the new religious affirmation. In such mosques the general direction of the carpets often makes an angle with the axis of the building. This is very striking for example in Aya Sophia.

During my stay in Tlemcen I was amazed to find that the maḥārīb were generally placed in the wrong direction—and this in mosques which had been built as such and in the construction of which much expense and care had been devoted. The miḥrāb of the great mosque of Tlemcen is oriented toward the south ; the same is the case for the mosque of Sīdī Bel Ḥassen, now used as a museum, and for the magnificent ruined mosque of Manṣūra. In Fās, I was told that the miḥrāb of the medarsa Seffarine had the best orientation, which implied that other maḥārīb were more or less deficient in that respect. In Cairo, I was told the same thing about the miḥrāb of al-Azhar.

Sometimes, it is true, the error is corrected. For example, in one of the Tlemcen mosques (I think it was in the great mosque) I saw a mark in the semicircular miḥrāb indicating the true qibla. I was told that the imām of the great mosque leads the prayers standing in a corner of the miḥrāb and shows by his position the holy direction. According to Prof. Snouck Hurgronje, in many mosques of the Dutch East Indies the qibla is indicated by the drawing of lines or the stretching of threads, and the faithful take their direction from these lines or threads, not from the miḥrāb itself. (2)

---

(1) See C. Schoy in Encyclopaedia of Islām (vol. 2, 987-9, 1927).
(2) Encyclopaedia of Islām (vol. 2, 986, 1927).

The laxity in the usual determination of the qibla is well illustrated by the Arabic language, for in the Maghrib that word and its derivative qiblī refers to the East, while in Syria and Egypt it refers to the South. On early Turkish rhomb-cards the south is called either al-janūb or al-qibla. It was probably because of that confusion that Eastern Muslims attached more importance to the southern end of the compass needle than to the northern one as we do (3). However, in Egypt, and presumably also in other countries, they sell little compasses indicating the true qibla.

Has a survey ever been made of the directions of the maḥārīb? It would be interesting to undertake it. That survey could not be made completely except by a Muslim, for many mosques are closed to infidels. I would like to know how correct the maḥārīb are in different countries, and at different periods; this would give us some indications with regard to the intellectual accuracy of the people of those times and places. The facts I have thus far been able to collect (outside of my own observations) are very meagre and divergent.

GEORGE SARTON.

Query no. 26.—*Le Prospectus de l'Encyclopédie* (1750).

Dans l'édition des Œuvres complètes de DIDEROT de JULES ASSÉZAT (1832-76) (20 vols., Paris, GARNIER, 1875-77) on trouve au vol. 13 (p. 129-64), le texte du *Prospectus* qui (d'après l'éditeur) parut en octobre 1750, daté 1751. Ce texte commence ainsi (p. 129) :

« L'ouvrage que nous annonçons n'est plus un ouvrage à faire. Le manuscrit et les dessins en sont complets. Nous pouvons assurer qu'il n'aura pas moins de huit volumes et de six cents planches, et que les volumes se succèderont sans interruption. »

Il diffère très sensiblement du texte qui fut placé par d'ALEMBERT à la suite de son « Discours préliminaire des éditeurs » dans le vol. 1 de l'Encyclopédie (1751). D'ailleurs D'ALEMBERT nous dit lui-même (p. xxxiii) :

« Après les réflexions & les vûes générales que nous avons crû devoir placer à la tête de cette Encyclopédie, il est tems enfin d'instruire plus particulierement le public sur l'Ouvrage que nous lui présentons. Le *Prospectus* qui a déjà été publié dans cette vûe, & dont M. DIDEROT mon collegue est l'auteur, ayant été reçu de toute l'Europe avec les plus grands éloges, je vais en son nom le remettre ici de nouveau sous les yeux du Public, avec les changements & les additions qui nous ont paru convenable à l'un & à l'autre. »

(3) See my *Introd.*, 2, 630.

Il ressort clairement de ce passage, si l'on en doutait encore, que ce *Prospectus* fut publié indépendamment et sans doute fut-il tiré à un grand nombre d'exemplaires, et cependant il semble avoir complètement disparu ! Des recherches que Madame METZGER a bien voulu faire pour moi à la Bibliothèque nationale de Paris n'ont donné aucun résultat. D'autre part cette Bibliothèque avait organisé cet été (1932) une exposition relative à l'Encyclopédie et autant qu'on en peut juger par le catalogue qui en a été publié (voir *Isis*, 19, 400), aucun exemplaire du *Prospectus* n'y a été montré. Il est possible toutefois que des exemplaires en existent encore ailleurs ? Peut-être certains se trouvent-il insérés dans le premier volume de l'Encyclopédie, sans être catalogués séparément ? Il vaudrait la peine de faire des recherches à ce sujet, et si un exemplaire original de ce *Prospectus* était découvert je serais heureux d'en publier une description illustrée dans *Isis*, avec l'indication des variantes principales par rapport aux textes donnés par D'ALEMBERT et par ASSÉZAT.

GEORGE SARTON.

II. — MISCELLANEA.

Paris.—*Institut d'Histoire des Sciences*, année scolaire 1932-1933. Cours des Facultés et des Etablissements publics d'Enseignement supérieur à Paris, coordonnant le programme de l'Institut.

M. ABEL REY, directeur de l'Institut : *Histoire de la Science hellénique à partir des temps platoniciens.*

Etudes d'Histoire philosophique des Sciences (Cours préparatoire au Certificat d'Histoire et Philosophie des Sciences).

M. LAIGNEL-LAVASTINE : *Histoire de la Tuberculose.*

M. LE ROY : *Le problème de l'atome, d'après la science et la philosophie.*

La portée philosophique des nouvelles données de la physique et de l'astronomie.

M. MASSIGNON : *La formation du vocabulaire scientifique arabe chez les encyclopédistes quarmates du Xe siècle* (IKWĀN AL SAFĀ).

M. MAUSS et M. SIMIAND : *Histoire des Techniques.*

Conférences de l'Institut :

M. P. BRUNET : *L'Astronomie physique de* 1687 *à* 1759.

M. MAROTTE : *La genèse de notre connaissance du système du monde, envisagée dans ses relations avec l'enseignement élémentaire de la cosmographie.*

M. VOLKRINGER : *L'évolution des théories sur la nature de la lumière.*

M. DAUJAT : *Les travaux de* DUHEM *sur les origines du principe de Pascal en hydrostatique.*

Madame METZGER : *La philosophie de la matière chez les chimistes du XVII*e *et du XVIII*e *siècles.*

M. BRAUMAN : *Histoire de la technique chimique jusqu'au début de la chimie pneumatique.*

M. LAIGNEL-LAVASTINE : *Les grandes étapes de la pensée biologique* (avec projections) : (Cours préparatoire au Certificat d'Histoire et Philosophie des Sciences).

Centenaire de la mort de Nicéphore Niepce.—Le 4 juin 1933 a été célébré à Châlons sur Saône, le centenaire de la mort de JOSEPH NICÉPHORE NIEPCE (7.03.1765 — 3.07.1833) qui a, le premier, obtenu en 1824, une photographie positive de l'image d'une chambre noire (*Isis*, 8, 511 ff.). L. S.

Reviews

Claudii Ptolemaei ***Geographiae Codex Urbinas Graecus*** **82. Phototypice depictus consilio et opera curatorum Bibliothecae Vaticanae (Codices e Vaticanis selecti quam simillime expressi iussu Pii XI P.M. Series maior, Volumen XVIII. Tomi quatuor, folio maiore et quarto). MCMXXXII Lugduni Batavorum apud E. J. Brill, Lipsiae apud Ottonem Harrassowitz (350 copies printed costing 450 Dutch florins each).**

Tomus prodromus. Josephi Fischer S.J. de Cl. Ptolemaei vita operibus geographia praesertim eiusque fatis. Pars prior. Commentatio (xvi+608 p.)

Tomus prodromus. Pars altera. Tabulae geographicae lxxxiii graecae arabicae latinae e codicibus liii selectae (xii p., plates).

Pars prior. Textus cum appendice critica Pii Franchi de Cavalieri (40 p.; facsimile).

Pars altera. Tabulae geographicae. Accedunt tabulae viginti septem codicis Vaticani latini 5698 tres Urbinatis graeci 83. (elephant folio volume; the three other parts being quarto).

This great work which marks the culmination of Father Joseph Fischer's long activity (1) is so important that it is worthwhile to describe its genesis. But first of all let us indicate in a few words how important it is. Ptolemy's geography, the *Γεωγραφικὴ ὑφήγησις*, is undoubtedly one of the greatest books in the whole history of science: it has deeply influenced geographical knowledge for more than twelve centuries. Now this is the first complete edition of that fundamental book, text and maps, together with a thorough introduction and an imposing apparatus ready for further study.

In the autumn of 1899 the author made a startling discovery in the library of Wolfegg castle (in Württemberg), one of the seats of His Serenissime Highness the prince of Waldburg-Wolfegg-Waldsee. In that

(1) See *Isis* passim. Yet the Fischer bibliography in *Isis* is very incomplete, partly because he sent us only a few reprints.

single library he found four cartographical unica : the two world maps of MARTIN WALDSEEMÜLLER (Ilacomilus) of 1507 and 1516, another and still larger world map by JODOCUS HONDIUS, 1611, and best of all, a Latin PTOLEMY Ms. of DONNUS NICOLAUS GERMANNUS (text and maps), dated 1468, which proved to be the prototype of the Ulm edition of 1486. Encouraged by this find, Father FISCHER began a systematic search in the map sections of many German and Italian libraries (1903-4) ; he found no new WALDSEEMÜLLER map but a rich collection of Ptolemaic Latin MSS. While he was working at the Ambrosiana in Milano he was helped by Mons. Dr. ACHILLE RATTI (now H. H. PIUS XI !) who drew his attention to a Ptolemaic Greek codex (Cod. Mediol. Ambr. graec. D 527 inf., alias gr. 997) including a world map, 64 provincial maps, and 4 general maps of Europe, Libya, N. Asia and S. Asia (the Latin MSS. contain only a world map and 26 provincial maps). Father FISCHER examined that codex and concluded that however interesting for the history of cartography, it was but of little use for the study of the early Ptolemaic tradition.

Pursuing his investigations he discovered in the Vatican the Codex lat. 5698 which was strikingly different from the other Latin MSS. and printed copies. Could there be a Greek prototype to that Codex? Yes, there was, and the indefatigable searcher found it in the same Vatican library : the Urbinas graec. 82. He also discovered at the same time that the Urbinas graec. 83 was closely connected with the Ambr. gr. 997 above mentioned.

He was thus led to distinguish two different Greek traditions : the A redaction (I use his own symbols) containing a world map and 26 special maps, and the B redaction, containing a world map and 64 (or 68) special maps. These redactions differ also in many other respects. Now all the Latin MSS. contain only the 27 maps of the A redaction, but the nomenclature and outline of these maps often show traces of the B tradition.

Father FISCHER was then already convinced of the superiority of Urbinas graec. 82, but for the sake of thoroughness he decided to investigate the English and French libraries (1909-10). He even examined the Syrian, Arabic, and Armenian maps, or rather this was done for him by Dr. HANS V. MŽIK. As the importance of his purpose and his own earnestness and ability were fully appreciated he received considerable help not only from his own people, the Jesuit institutes of Maria-Laach and Feldkirch, but also from almost all the leading historians of cartography. During a third journey of exploration (Rome, 1925) he investigated such questions as the relationship between Urb. graec. 82 and the MS. of the Vatopedi monastery of Mt. Athos ; the relationship between

the text of Urb. graec. 82 and its maps ; the relationship between the Urb. world map and the additions ascribed to AGATHODAEMON (2) ; etc.

Thanks to combined efforts of Father FISCHER and of many other scholars we may now consider established (as well as anything of that kind can be) that PTOLEMY was the real author of the text and the originator of the maps.

In his study of the Greek *text* C. MUELLER had already reached the conclusion that there were two traditions of that text, which he called the Byzantine and the Asiatic tradition. This corresponds roughly to the A and B redactions of Father FISCHER ; and Urb. graec. 82 (of c. 1200) is our most ancient representative of the Byzantine or A tradition.

MÜLLER had neglected the cartographic evidence on the assumption that the maps could not be traced back to PTOLEMY. In fact the maps are as old as our oldest MSS., and we know that PTOLEMY's very aim was the production of a world map, witness his statement : *Προκειμένου δ'ἐν τῷ παρόντι καταγράψαι τὴν καθ' ἡμᾶς οἰκουμένην σύμμετρον ὡς ἔνι μάλιστα τῇ κατ' ἀλήθειαν* (3) (I, 2, 2).

The thorough study of all the maps available convinced the author that the maps of Urb. graec. 82 should be taken as the basic maps. They are remarkably well completed by the maps of Vat. lat. 5698, both with regard to nomenclature and outline. Unfortunately in the present publication the corresponding Latin and Greek maps are not opposite one another and minute comparison between them is very difficult if not impossible,—unless the book be unbound and the maps loosened.

Father FISCHER's great undertaking was much delayed by the war and the printing of it suffered many interruptions, with the result that it was necessary to print abundant additions in the end (p. 515-54 of Prodromus). The most important of these additions refer to various Constantinople MSS. communicated by Dr. ADOLF DEISSMANN. Other notes were suggested by the writings of ALBERT HERRMANN, ERNST HONIGMANN (*Isis*, 14, 270-76), PAUL SCHNABEL, ALBERT REHM, PAUL DIERCKE, OTTO CUNTZ, etc. Some of the notes (as well as parts of the main text) were contributed by Monsignore G. MERCATI, who took a very deep interest in the work throughout. To be sure it would have been more convenient if the substance of these addenda could have been integrated into the body of the work, but it is better to have them as they are than not at all.

---

(2) FISCHER, 1916; *Isis*, 5, 206; *Introduction*, I, 276.

(3) Latinized by MUELLER as follows : *Sed quum in praesentiarum propositum sit, nostram terram habitatam ita describere, ut delineatio quam maxime congruat cum vera terra.*

The Arabic maps in the second volume of the Prodromus (Cod. Const. arab. 2610 ad S. Sophiae) are borrowed from the Monumenta cartographica Africae et Aegypti edited by the Egyptian Prince YOUSSOUF KAMAL (vol. 1, Cairo, 1926, etc., *Isis*, 12, 393), a book which I have not yet succeeded in seeing.

The present work is fundamental but so expensive that it is beyond the reach of any but the very largest libraries. Moreover, it is not easy to handle, and requires further study. Let us hope that some scholar, taking full advantage of it, will give us an English survey of Ptolemaic geography and of all the problems relative to it, and publish it in a convenient shape and at an accessible price. In any case, this work will remain fundamental—an eternal monument to the author's memory.

GEORGE SARTON.

Ptolemy.—*Geography of Claudius Ptolemy*. Translated into English and edited by EDWARD LUTHER STEVENSON. Based upon Greek and Latin manuscripts and important late fifteenth and early sixteenth century printed editions. Including reproductions of the maps from the Ebner Manuscript, ca. 1460. With an introduction by JOSEPH FISCHER, S.J. XVI+167 p., 29 facsimile maps. New York, Public Library, 1932 (250 copies).

When I was reviewing a few weeks ago FISCHER's great Ptolemaic edition and commentary (4 vols., Leide, 1932) and expressing the wish that an English version of the Geography might be available, I did not realize that my wish would be so promptly fulfilled. And yet I should have known that Dr. STEVENSON was engaged in that undertaking, for I have followed his fruitful activities as closely as he would allow me (see, e.g., *Isis*, 4, 549-53, including an imposing list of his earlier publications.)

The present splendid volume is appropriately published by the New York Public Library because it contains a full size facsimile reproduction of the 27 maps of the Codex Ebnerianus which is now one of the main treasures of that library. To these maps have been added two others: the Ruysch map of the printed Ptolemy of 1508, and the New World map of the printed edition of 1522, bearing the word "America" printed across S. America, where WALDSEEMÜLLER had placed it in his World Map of 1507.

The New York Public Library might have restricted itself to the publication of this valuable facsimile, but it considered with reason that no facsimile publication is justified without explanatory material, and added to it a complete English version of the text of the Geography (the first complete version in any modern language), together with an

English summary of FISCHER's learned *Commentatio* (published in the Tomus prodromus of his Ptolemy above-mentioned). I shall return to this presently.

To underline the complexity and difficulty of the investigations concerning PTOLEMY's geography, it will suffice to say that there are more than fifty printed editions of it, and more than forty MSS. (Greek, Latin, Arabic) though very few of these are fairly complete. Of the complete MSS. five are now available in facsimile editions, to wit:

(1) Codex Athous graecus, from the Vatopedi monastery on Mt. Athos. Published by VICTOR LANGLOIS, Paris, 1867.

(2) Ms. latin 4802, Bibliothèque nationale, Paris. Latin translation of JACOPO D'ANGIOLO. Published by H. OMONT (Paris, 1925) (not seen).

(3) Codex Urbinas graecus 82. Published by Father FISCHER, 1932.

(4) A DONNUS NICOLAUS GERMANUS MS. issued in a limited edition, photographed and hand colored. Known to me only by STEVENSON's mention of it in his own preface (p. XVI). He says that it is one of the finest MSS. known but does not state which it is and where it is (?!).

(5) Codex Ebnerianus reproduced in the present volume.

It should be noted at once that this last named Latin MS. dates from c. 1460 while the Urbino Greek MS. reproduced by Father FISCHER dates from c. 1200.

The Introduction provided by Father FISCHER to this volume is a summary of the very elaborate introduction to his own. A great part is devoted to the proof of the originality of the Ptolemaic maps. Father FISCHER's conclusions may be quoted (p. 13):

"He who assumes that the text of PTOLEMY goes back to PTOLEMY himself, must assume also, in accordance with the principles of a sound criticism, that the maps similarly transmitted also go back to PTOLEMY. If we inquire more carefully about the relation between maps and text three interpretations are possible: (1) the maps might have been drawn according to PTOLEMY's text; (2) the text might have been derived from the maps; (3) or both text and maps may show an independent transmission, so that the maps may represent the remodeled and completed MARINUS maps, and the text the remodeled MARINUS text. All three possibilities I have investigated thoroughly. In this investigation I came to the conclusion that the third possibility conforms to the truth (*Commentatio*, p. 136-158). Still another legitimate objection is, however, to be met: If PTOLEMY had actually added to his Geography the maps transmitted to us, then traces of them would necessarily be found in the literature of the ancient people. In my *Commentatio* I have treated in detail the question whether or not from the traditions of the subsequent time the existence of PTOLEMY's maps can also be proved. The investigation revealed that the maps of PTOLEMY exercised, for a long time, influence among the Greeks and Byzantines (AGATHODÄMON, PAPPUS, MARCIAN OF HERACLEA); among the Syrians, Arabians and Armenians (JACOBUS OF EDESSA, AL-KINDI, MUHAMMAD IBN MUSA AL-KHWARIZMI, Pseudo-MOSES OF CHORENE); among the Romans and the Germans (JULIUS AFRICANUS,

AMMIANUS MARCELLINUS, JACOBUS ANGELUS, FRANCESCO DI LAPACINO, DOMENICO BUONINSEGNI, DONNUS NICOLAUS GERMANUS, HENRICUS MARTELLUS GERMANUS, MARTIN WALDSEEMÜLLER); *Commentatio*, p. 417-490."

However, the case of the Prolemaic World Maps is different (p. 5) :

" It is a strange caprice of fate that just that map of the world which one so often sees copied as PTOLEMY's map of the world does not go back to PTOLEMY. That it does not show the modified spherical projection which, according to PTOLEMY's plain words (at the end of the last chapter of Book I), was certainly to be expected, but the simple conical projection, SCHÖNE has already correctly noted and for this reason has justly denied that PTOLEMY could be regarded as the originator of the world map. In the meantime ADOLF DEISSMANN has succeeded in finding in the Serail Library of Constantinople, a Ptolemaic manuscript in Greek which gives the world map in the modified spherical projection as used by PTOLEMY. Professor DEISSMANN kindly placed at my disposal text and maps of this manuscript in photographic reproduction. It was my first impression that I had before me the genuine world map of PTOLEMY, but I was soon convinced that although the projection is that of PTOLEMY the contents of the map do not correspond at all to PTOLEMY's data as they are found in the last three chapters of Book VII, but have been greatly modified by the additions of AGATHODÄMON (*Commentatio*, p. 515-521). In the final results of my earlier investigations nothing therefore is changed by the new discovery; the twenty-six provincial maps of the A-redaction, and the sixty-four small provincial maps of the B-redaction go back to PTOLEMY, but the world map, essentially the same in both the A-redaction and the B-redaction, is to be accredited to the Alexandrian geographer AGATHODÄMON."

What was PTOLEMY's problem? (p. 7) :

" PTOLEMY wanted to restore completely, not only a map which would correspond, as far as possible, to the existing text of the latest edition of MARINUS, but he wanted also to furnish an improved MARINUS map altogether according to the ideas of MARINUS himself. In order to attain this end and secure it for the future, he had to solve, in addition to his main problem, two secondary problems. He had (1) to submit the collected material to a reëxamination, and (2) to perfect the map-drawing (a) by an exact and tabular presentation on one plane, not on eight or more, and together the positions in longitude and latitude of that which was to be entered, and (b) by determining a proper kind of projection. How PTOLEMY solved these secondary problems we learn from chapters six to twenty-four of Book I, and in detail in Books II-VII."

The Latin Ms. here reproduced is important because it was the original copy for the Roman printed editions of 1478, 1490, 1507, 1508 in which the Ptolemaic maps were reproduced more accurately than in others, yet the reader must be warned that DONNUS NICOLAUS introduced changes, namely in the projection. Let us quote again Father FISCHER (p. 10) :

" Instead of the modified projection of MARINUS, selected by PTOLEMY for his territorial and provincial maps, in which only the middle parallel of the special

map was divided in right proportions, DONNUS NICOLAUS (who is erroneously called DONIS NICOLAUS in the Ulm edition of PTOLEMY of 1482) has on each map divided at least two parallels in right proportion. Thereby he has attained effectively the greater similarity striven after by PTOLEMY, with a drawing upon the globe. The maps of the reproduced Codex Ebnerianus plainly show this. Since the ' DONIS Projection ' introduced by DONNUS NICOLAUS GERMANUS became generally known (FISCHER, *Die Entdeckungen der Normannen in Amerika*, Freiburg, 1902, p. 82, Beilage 6; English edition, London, 1903, p. 79) in the Rome editions of 1478, 1490, 1507, and 1508, the Ulm of 1482 and of 1486, the Strassburg of 1513, 1520, 1522, and 1525, it has long been considered as the actual Ptolemaic one, and later on, just this very projection has had to serve as proof that the maps of PTOLEMY could not have originated with PTOLEMY, since he assuredly would not have applied the ' DONIS Projection,' but the modified MARINUS projection. We see by this example to what conclusions one is led, unless one goes back to the original source."

PTOLEMY's Geography can never be given too much importance, for it was the fountain source of geographical knowledge for more than twelve centuries. The more reason for pointing out its defects. STEVENSON does it briefly and clearly as follows (p. xv) :

PTOLEMY " greatly exaggerated the total longitude of the inhabited earth, and yet he reduced this from that given by MARINUS and by others who had preceded him. He increased by almost one third the length of the Mediterranean; he makes the Indian Ocean an enclosed sea by joining the south-eastern region of Asia to southern Africa, and by those who accepted his geography this might well have been the reason for less vigorous and less early effort to reach the Indies of the East by an attempt to circumnavigate Africa; he increased very greatly the size of the island Ceylon."

The facts given by me in my *Introduction* (vol. 1, 272) should be corrected as follows. PTOLEMY's lifetime extended from about 90 to 168. The place of his birth remains uncertain (I said " Egypt," but even that is more than we know). It was perhaps Ptolemais Hermeiou, on the Nile below Abydos ; but not Pelusium, which name rests on a misunderstanding. ABŪ-L-WAFĀ' description of PTOLEMY's personal appearance, well known through the translation of GERARD OF CREMONA in the foreword to the *Almagest* edition of 1515, is interesting but in spite of its engaging concreteness unreliable. His moral character is better known through his own utterances, chiefly this one in the *Almagest*, book I (p. 14) :

" We have come to believe that it is our duty, on the one hand, to regulate our actions harmoniously in order that we ourselves, in the contingencies of daily life may never forget to pay regard to noble demeanor and tactful bearing; on the other hand to devote our entire strength to intellectual activity, for the purpose of imparting instruction in theoretical knowledge, whose branches are numerous

and glorious; preëminently, however, to give instruction in that realm which is specifically comprehended under the name of Mathematics.' "

I shall not attempt to judge STEVENSON's English version as this would require a long and painful comparison with the Greek text, but it is clear that his task, however well done as far as it goes, is still incomplete in many respects, and thus this very beautiful volume does not yet fulfill our hopes after all.

To begin with, this English version being a part of an expensive and rare publication will only be found in relatively large libraries, and even there it will be kept under lock and key among the more precious volumes and will not be available to students except with many restrictions.

What we still need is an English version (possibly this one) printed in a volume of moderate size and price, together with a study of the Ptolemaic MSS. and editions, of PTOLEMY's life and works, and of the problems connected with MARINUS and AGATHODAEMON.

The English version and the Greek text should be printed on opposite pages, the main variants of the fundamental MSS. being quoted (there are no variants whatever in STEVENSON's book !). Modern coordinates should be printed in tabular form alongside of the Ptolemaic latitudes and longitudes, and other geographical references should be added to the text. Of course the main difficulty would be to restrict oneself to the essential. The task would be very difficult indeed, but it is so important that it would deserve to engage the most careful efforts of the most intelligent scholar. Finally there should be an index. The maps might be reproduced in cheaper editions (1), and it would be easy to index them by means of a sheet of transparent paper divided into numbered squares, such a sheet being inserted into each volume. In any case the text and maps should be published in two separate volumes, in order that both may be consulted simultaneously, and in different formats, the large size needed for the maps being very inconvenient for the text.

All of which shows that there is still plenty of work reserved for future scholars, but we must be thankful to Dr. STEVENSON for having brought us a little nearer to the goal.

GEORGE SARTON

The Mishnat ha-Middot, ***the first Hebrew Geometry of about*** **150** ***C. E.,***
and the Geometry of **MUHAMMAD IBN MUSA AL-KHOWARIZMI,** ***the***

(1) The British Museum has shown how this could be done beautifully yet cheaply (*Isis*, 13, 162, 236).

first Arabic Geometry** (c. 820), **representing the Arabic Version of the Mishnat ha-Middot. A new edition of the Hebrew and Arabic texts with introduction, translation and notes by SOLOMON GANDZ, with 14 figures in the text and 4 plates (*Quellen und Studien zur Geschichte der Mathematik, Astronomie und Physik*, herausgegeben von O. NEUGEBAUER, J. STENZEL, O. TOEPLITZ, Abteilung A : Quellen, 2 Band). JULIUS SPRINGER, Berlin, 1932. Paper, pp. VIII+96.

The text of the *Mishnat ha-Middot* was first discovered by MORITZ STEINSCHNEIDER in 1862 in MS. 36 of the Munich Library, and it was published by him for the first time in 1864 in honour of the seventieth birthday of LEOPOLD ZUNZ, the father of Jewish science. In 1880 the text was again edited and published by the distinguished mathematician, HERMANN SCHAPIRA, who was the first to perceive the extraordinary likeness between the *Mishnat ha-Middot* and one section of the algebra of MUHAMMAD IBN MUSA AL-KHOWARIZMI, one of the earliest of Muslim mathematicians. The editor of the present edition has sought not only to emend and occasionally even to reconstruct the Hebrew text by the help of a new fragment of the *Mishnat ha-Middot* discovered by NEUGEBAUER in the Bodleian Library, Oxford, and, being an Arabic scholar which SCHAPIRA was not, by comparison with the treatise on mensuration of AL-KHOWARIZMI, but also to elaborate and establish conclusions which SCHAPIRA, with no knowledge of Arabic and not so versed as the present editor in the history of Oriental mathematics, could only surmise.

A book named *Mishnat*, *Baraita*, or *Midrash* of forty-nine *Middot* is cited by such early Jewish authorities as RASHI and IBN EZRA, but such quotations as these give, are of a Midrashic nature, dealing with measurements of the tabernacle, or exegesis of the Bible, and many of them have been found in the *Baraita de Meleket ha-Mishkan*. The *Mishnat ha-Middot* discovered by STEINSCHNEIDER, on the other hand, is a treatise on mensuration with only forty-two paragraphs and containing none of these quotations, although there appear in it quotations found by E. CARMOLY in a commentary of COMTINO to the *Yesod Mora* of IBN EZRA and a commentary of SHABBATAI BEN MALKIEL to the *Sepher ha-Shem* of IBN EZRA, and COMTINO states that the forty-nine *Middot* are based upon the science of mensuration and geometry. The identity or non-identity of the two books is a difficult question, and the value of COMTINO's evidence would bear further investigation, especially his statement concerning the number of paragraphs, since that may be a crucial point in the problem.

Professor GANDZ accepts the identity of the two books, basing his judgment on new facts supplied by the Bodleian fragment. For it has, in addition to the five chapters of the Munich MS., a sixth chapter

which is entirely Midrashic in nature and which contains some of the quotations made by RASHI from the *Mishnat* of the forty-nine *Middot*. GANDZ supposes, therefore, that " the old *Mishnat ha-Middot* contained two parts, a Midrash to the chapter of the Bible dealing with the construction of the tabernacle, and a short compendium of geometry as a fitting introduction to this Midrash," but he is not prepared to stress the remarkable fact that the additional paragraphs brought or indicated by the Bodleian fragment would make the number of paragraphs in the old *Mishnat ha-Middot* exactly forty-nine, since, in his opinion, it is improbable that the text of the Midrash would consist of five paragraphs only and the geometrical introduction of forty-four. He assumes, therefore, that the title, " Forty-nine Middot," refers to the number of paragraphs in the Midrashic text, which would be represented today by the *Baraita de Meleket ha-Mishkan*. Since, however, COMTINO lived in Constantinople in the second half of the fifteenth century,and the Munich MS. was written there in 1480, it may be urged that his statement concerning the number of paragraphs is based on a version similar to that represented by the Bodleian fragment, and this, if true, would be of some consequence, since, then, the only link between the mathematical text of the *Mishnat ha-Middot* and the *Midrash* of the forty-nine *Middot* would be the sixth chapter supplied by the Bodleian fragment with RASHI's quotations of the forty-nine *Middot* in it, and it would be just as probable to consider the sixth chapter an adventitious adjunct to the mathematical treatise as to regard it as the first chapter of a Midrashic book. In any case it is quite conceivable that the two books were once separate entities and then were joined together for some purpose of the schools, either, as GANDZ suggests, as mathematical introduction and Midrashic treatise, or, what is also possib e, as a mathematical treatise with a Midrashic conclusion.

STEINSCHNEIDER dates the *Mishnat ha-Middot* in the early Arab period (c. 800-1200), led to this conclusion apparently by the scientific nature of the contents and the fact that a few of the mathematical terms correspond to Arabic terms. SCHAPIRA has shown, however, that the mathematical terminology of the *Mishnat ha-Middot* is different from that of the Hebrew mathematicians of the Arab period and holds that it must have been written earlier, most probably in the time of the *Mishnah*. GANDZ accepts SCHAPIRA's view and seeks to confirm it by a careful comparison with the treatise on mensuration of AL-KHOWARIZMI, in which he endeavours to demonstrate that AL-KHOWARIZMI is dependent on the *Mishnat ha-Middot*. His thesis is that in the *Mishnat ha-Middot* we have " a genuine treatise of an old Mishnah," and that it forms " a missing link in the transition of non-Greek, Oriental mathematics."

The date which GANDZ advances for the *Mishnat ha-Middot*, however, about 150 C.E., is based on two considerations, in the first place, the vocabulary, diction, style, and terminology of the treatise, and secondly, the authorship. The present writer feels himself incompetent to judge of the first consideration and would only remark that vocabulary, diction, style, and terminology, are not always dependable indications of date. The crux of the matter lies in the authorship, and it should be pointed out that the name, NEHEMIAH, occurs only twice, and both times in the Bodleian fragment only, once in the Midrashic sixth chapter and again in the fifth, where the same mathematical formula is attributed to NEHEMIAH as in the sixth chapter, a chain of circumstances calculated to give rise to suspicion. Moreover, the connexion of this name with the Rabbi NEHEMIAH of the second contury C.E. is, of course, a conjecture, resting for the most part on the fact that he appears to have been interested in mathematical computations.

The whole literary and historical background of the problem presented by the book is discussed by GANDZ with great acumen and scholarly simplicity in his introductions to the Hebrew and Arabic texts, and the problem is laid bare in such a masterly fashion and the facts stated so candidly that it is possible for a scholar to draw his own conclusions, if he does not agree with those of GANDZ. The emendations and reconstructions of the Hebrew text proposed by GANDZ are the fruits of ripe scholarship and based on genuine philological grounds, many of his notes are nothing short of essays on the historical development of mathematical terminology, and so far as the texts and translations are concerned, the edition is as definitive as can well be expected. In the statement of his thesis, however, there is some confusion, and the evidence on which he relies to demonstrate it, will not be accepted in toto without further proof.

On page 63 GANDZ writes that " the comparative table on p. 85 makes it evident that the bulk of AL-KHOWARIZMI's geometry was *verbally* taken from the *Mishnat ha-Middot*," and on page 64 that it is " an almost *verbal* translation of an old Hebrew geometry," and yet on the same page as the latter citation we are told that " AL-KHOWARIZMI's work and life offer no further evidence for the assumption of a familiarity with Hebrew language and thought " and that " there is more probability for the conclusion that this Hebrew geometry was available in a Persian or Syriac translation." The full implications of this last statement have not been duly weighed by GANDZ, otherwise he would not have talked of a verbal translation, nor would he have asserted so often in his notes that AL-KHOWARIZMI is translating in this or that place Hebrew words or phrases.

Moreover, the comparative table on page 85 does not prove that the bulk of AL-KHOWARIZMI's geometry was taken from the *Mishnat ha-Middot*. The order of the sections is entirely different. In one section the Arabic has another text altogether, and another section is not represented in the Hebrew at all. Sometimes the Hebrew is fuller, at other times the Arabic. In some sections the Arabic arranges the material quite differently from the Hebrew, in others it adds proofs that appear to be of a more developed type than those given in the *Mishnat ha-Middot*, not to speak of phrases and sentences that are occasionally of vital import and which GANDZ on two occasions at least (Cf. p. 29, note 38) inserts into the Hebrew text with no other justification than that the author of the *Mishnat ha-Middot* shows in another section that he knew the required formula, a plausible argument, if we overlook the fact that the *Mishnat ha-Middot* has probably had a history of its own. These facts do not point to a direct dependence of the one book upon the other, but only to a family resemblance, and AL-KHOWARIZMI's chapter on mensuration is probably a more advanced type of a common school text, of which an earlier type served as basis for the *Mishnat ha-Middot*.

Direct dependence of the Arabic text upon the Hebrew might be granted, if GANDZ's assertion on page 7 that " AL-KHOWARIZMI's geometry shows terms and phrases which seem more familiar and congenial to the Hebrew than to the Arabic language, " could be substantiated, but none of the instances which he offers in his note on that page can be accepted. *'Amūd*, for example, is used by AL-KHOWARIZMI's older contemporary, AL-ḤAJJAJ, in his translation of EUCLID. So also is *dhira' fi dhira'* and *ḍaraba fī*. *'Umq* is the ordinary Arabic term for depth, and *misāḥah* occurs only five times, so far as I have noticed, three times in the text and twice in figures, and its meaning in one case at least is clearly not area, for which the term generally used is *taskir*, but measurement, as GANDZ himself translates it in this place, and which is a quite possible translation in the other instances also. With respect to other terms referred to in the notes to the text, *saṭḥ*, *watar*, *dil' jadhr*, and *quṭr*, are all to be found in AL-ḤAJJAJ, and *quṭr* does not mean " dropping-rain," as GANDZ supposes on page 15, note 16, the Arabic term for which is *qaṭr*. That *Qā'im* is a translation of the Hebrew *nizẓābhā* is far-fetched, since in the Arabic *Qā'im* is only the half of a phrase, which is represented in the Hebrew by the one word *nizẓābhā*. The one case which is worth investigation is that of *qaus* (segment), which GANDZ derives from the Hebrew *qeshītha*, and for which the usual Arabic term is *qi'ṭah*, but the name, bow, is such a natural one for a segment that the claim of dependence in such an instance is a mere conjecture ; and it should be observed also that AL-KHOWARIZMI begins this section

with the statement that " each segment (*qiṭ'ah*) of a circle is like to a bow (*qaus*)," using the bow as a picture, so-to speak, of a segment.

GANDZ's notes about the term, root, are also somewhat puzzling. AL-KHOWARIZMI writes, section 1. f., p. 68, " For any square area with equal sides one of its sides multiplied by one is its root." GANDZ translates, " multiplied by one [square unit]," and on page 37, note 11, he asserts that the same definition of root is given by ABU KAMIL SHUJA and refers to KARPINSKI in *The American Mathematical Monthly*, XXI (1914), p. 40, where we find the following statement of ABU KAMIL : " Of that square any side multiplied by *one in number* is the same as the area of a surface whose length is the side or the root of the square." The origin and meaning of the phrase, " square unit," is, at least to the present writer, not clear, and there is no support for it in either the present Hebrew or Arabic text.

Two remarks in conclusion. The fact that AL-KHOWARIZMI's book on mensuration shows little or no sign of influence from the side of Greek theoretical mathematics does not prove either his ignorance or his dislike of that mathematics. The only legitimate inference is that he did not use it, or find it useful, for his purpose, a fact that can be explained by the practical nature and object of the book and perhaps by the traditions of the schools. For although AL-KHOWARIZMI probably lived some of his life far away from Greek countries, as GANDZ says, he spent a considerable part of it nevertheless in a country where the Nestorian schools still flourished and where AL-ḤAJJAJ translated EUCLID.

I would like also to suggest that it would have been better if GANDZ had relegated all emendations and reconstructions of the Hebrew text with their translations to the notes, especially those based on AL-KHOWARIZMI, for even although he carefully draws attention to all such changes in his notes, the method he has used is apt to confuse some of the issues, and even the best of proof-reading is liable to error, as page 40 will prove, where there should be emended text within square brackets according to note 13.

The following errors have been observed. P. 14, note 12, read " roof " for " root,"; p. 15, l. 3, read " in its length " for " in the length "; p. 15, note 16, read " Arabic " for " Arabiq " ; p. 25, l. 8, read " by " for " upon " ; p. 26, note 16, read " ist " for " sti " ; p. 27, note 22, read she-hī' for shīh'; p. 29, l. 8, read ha-dōr for ka-dōr ; p. 38, note 3, add " of " after " instead " ; p. 43, l. 17, read " the trilateral all of whose sides are equal " for " the equilateral, each side (multiplied) into itself " ; p. 50, last line, read " by " for " upon " ; p. 68, note 8, read " distinction " for " distinktion " ; p. 74, l. 1, read z for r twice ; p. 76, l. 14, read yakhruju for yakhruḥu ; p. 77, l. 27, read " angle " for " ange " ; p. 79, l. 21,

read " base " or " basis " ; p. 82, l. 14, read " into " for " in " ; p. 96, read mutawāzi for mutawādhi.

Harvard University. WILLIAM THOMSON.

Baron Rodolphe d'Erlanger.—*La musique arabe.* Tome 1. AL-FĀRĀBĪ, ABŪ N-NAṢR MUḤAMMAD IBN MUḤAMMAD IBN TARKHĀN IBN UZLAGH. Grand traité de la musique. Kitābu-l-mūsīqī al-kabīr. Livres I et II. Traduction française. XXVIII+329 p., 125 fig. Paris, GEUTHNER, 1930.

Nous saluons avec plaisir la publication du premier volume de l'œuvre monumentale à laquelle le Baron d'ERLANGER, aidé du Baron CARRA DE VAUX et de MM. 'ABD AL-'AZĪZ BAKKOUCH et AL-MANNŪBĪ AS-SANŪSĪ s'est consacré depuis de longues années. Cette œuvre ne comprendra pas moins de sept volumes : les vols. 1 à 4 contiendront la traduction française des traités arabes fondamentaux ; le vol. 5, un résumé synthétique de ces traités ; le vol. 6, les règles de la musique arabe moderne ; le vol. 7, des exemples musicaux.

Cette œuvre est à rapprocher de celle de HENRY GEORGE FARMER, qui a été souvent discutée dans *Isis* (voir surtout vol. 8, 508-11 ; vol. 9, 559 ; vol. 13, 375-76 ; vol. 15, 370-72). M. FARMER prépare une édition critique et une traduction anglaise des traités musicaux arabes. Il a exposé quelques unes de ses vues sur la transmission des idées musicales dans notre revue : Greek theorists of music in Arabic translation (*Isis*, 13, 325-33, 1930).

Pour en revenir à l'œuvre de M. D'ERLANGER, son premier volume contient la traduction des deux premiers livres du traité d'AL-FĀRĀBĪ, plus une introduction générale, et une préface de M. CARRA DE VAUX. L'introduction ou " Avertissement " est assez maigre. L'auteur nous y donne une liste des auteurs qui seront représentés dans sa collection : AL-FĀRĀBĪ, IBN SĪNĀ, ṢAFĪ AL-DĪN et un commentateur inconnu de son *Kitāb al-adwār*, un compilateur anonyme du milieu du XV^e^ siècle, enfin 'ABD AL-ḤAMĪD AL-LĀDHIQI (1512). Quelques autres traités, par exemple ceux d'AL-KINDĪ et des Frères de la pureté ne seront pas inclus parce qu'ils ont déjà été traduits. Pour la traduction du grand livre d'AL-FĀRĀBĪ, l'auteur a fait usage de quatre MSS. : Leyde, Milan, Madrid, Beyrouth, dont les deux meilleurs (les deux derniers) sont malheureusement incomplets.

La préface de M. CARRA DE VAUX est à la fois fort instructive et suggestive. Qu'il me permette d'en citer quelques fragments (p. vii) :

« Cette théorie musicale, en principe, venait des Grecs; les savants arabes connaissaient plusieurs musicologues grecs; le traité de PTOLÉMÉE en particulier avait été traduit dans leur langue; mais FARABI a donné à la théorie un cachet

tout à fait personnel; il l'a clarifiée et approfondie; elle est devenue chez lui plus didactique dans certaines parties, plus analytique dans d'autres. L'ampleur de la composition, la pénétration de l'analyse, le tour philosophique et subtil de la pensée s'alliant à une grande expérience pratique, font de cette œuvre l'une des plus remarquables du moyen âge. La musique enseignée dans ce grand ouvrage paraîtra peut-être à certains lecteurs bien pauvre : elle est purement mélodique. D'harmonie, je veux dire d'accord entre les notes jouées simultanément, il n'est jamais question. Tout roule sur un instrument, assez modeste, en somme, le luth *(al-'ūd)*, instrument à quatre ou cinq cordes ne dépassant guère le champ de deux octaves. C'est sur cet instrument si simple et sur d'autres plus primitifs encore, les *tunbours* à deux cordes, que le théoricien exerce toutes les ressources de sa science, toute l'acuité de son analyse. »

Les peuples anciens — Hébreux, Egyptiens, Grecs, Persans — utilisaient plus d'instruments, et des chœurs nombreux · ils ont dû développer une sorte d'harmonie. Mais l'Islam découragea les efforts musicaux, et n'admit dans le culte que le mélopée de l'*adhān*, l'appel à la prière, traditionnellement attribuée à un Abyssin nommé BILĀL.

« La musique arabe se trouva donc réduite, en fait et en théorie, à la mélodie. Mais combien de ce côté elle se racheta; combien elle s'enrichit et s'affina ! Toute l'ingéniosité du théoricien, toute la sensibilité, toute l'inquiétude de l'artiste portent sur la mélodie. On arrive à employer des intervalles extrêmement ténus, plus petits même que le quart de ton; on distingue quantité de genres, des forts, des faibles, des *colorés*, et plus de vingt modes dont douze sont souvent employés; on varie; on nuance; on déplace certaines touches de quantités minimes; on ajoute des notes supplémentaires aux notes essentielles, des fioritures, des accents; on parvient à une subtilité merveilleuse. Avec si peu de moyens (1) la musique produisait sur les Orientaux une impression énorme..." (p. ix).

M. CARRA DE VAUX indique ensuite l'intérêt mathématique du *Kitāb al-mūsīqī*, mais il use d'une façon de parler assez maladroite lorsqu'il dit : " Les logarithmes sont là en puissance..." Il est à craindre que des lecteurs superficiels n'en concluent que les Arabes avaient en quelque sorte anticipé cette découverte. A la suite d'une de mes conférences sur la science arabe à Beyrouth l'un des auditeurs vint me dire que ses ancêtres avaient découvert les logarithmes. Peut-être avait-il lu cette préface ? Les logarithmes sont là comme la grammaire est dans le langage, ou comme les séries de FIBONACCI dans la structure d'êtres vivants(2), pas autrement. Mais continuons :

« L'école pythagoricienne et néoplatonicienne voulut donc mettre la musique en nombres. Ne disposant pas de moyens pour mesurer les vibrations, elle se servit

(1) D'après Mr. FARMER ces moyens étaient beaucoup plus riches que M. CARRA DE VAUX ne l'indique. Voir sa belle synthèse " Music " dans *Legacy of Islam* (Oxford, 356-75, 1931).

(2) Voir mon *Introduction*, 2, 611.

de longueurs de cordes; mais, gauche encore dans le maniement des fractions, elle commit dans son système quelques abus. La racine carrée lui fit peur; ainsi pour le demi-ton qui, le ton étant 9/8, devrait être $(\frac{9}{8})^{\frac{1}{2}}$ FARABI rejette cette racine; il partage le ton en deux rapports fractionnaires inégaux 18/17 et 17/16. Ces deux rapports sont de la forme $1 + 1/n$; c'est ce que nous appelons *rapports superpartiels* et que les Grecs avaient appelé *rapports épimères*. La théorie musicale a donné à ces rapports une importance véritablement abusive. Constatant que l'octave, la quarte et la quinte (2/1, 4/3, 3/2), puis le ton (9/8), qui sont les rapports fondamentaux, avaient cette forme, elle voulut que, pour tous les rapports de la même forme, les notes fussent consonantes (mélodiquement bien entendu); et cette prétendue loi, souvent invoquée, se trouva en contradiction avec le jugement de l'oreille. En physique on remarquera quelques tentatives pour créer une théorie de vibrations, et, à propos du passage de l'air dans les flûtes, de délicates analyses laissant entrevoir l'idée de la dynamique des fluides. Signalons en outre dans le traité de AL-FARABI et dans le commentaire à ṢAFIYU-D-DIN, de bons passages sur l'anatomie des organes de la voix ». (p. x).

Toute personne intelligente lisant ce traité (ou essayant de le lire) ne peut manquer de se demander : Quels rapports avaient toutes ces abstractions assez rebutantes avec la pratique musicale ? Malheureusement cette dernière est si mal connue qu'il est impossible de résoudre ce problème. Il est fort probable cependant que les musiciens — les vrais musiciens, les musiciens musiquant — n'étaient pas plus savants au X^e^ siècle qu'au XX^e^. D'autre part il est certain que la musique fut pratiquée en Arabie avant l'introduction des théories grecques. " Il se peut donc qu'au temps d'AL-FARABI la musique savante ait pris le cachet hellénique, en même temps qu'une musique purement arabe, au milieu d'influences variées, subsistait dans le peuple." (p. xiii).

On peut se demander aussi " D'où viennent les modes de la musique arabe actuelle ? " Mais les modes sont-ils vraiment inventés ; ne se développent-ils pas spontanément, inconsciemment, comme la syntaxe et la prosodie ? Nous attendons impatiemment les réponses que M. D'ERLANGER nous donnera lui-même dans son sixième volume, car il est à la fois théoricien et praticien, un compositeur distingué qu'un long séjour en Tunisie a familiarisé avec la musique d'orient.

On sait qu'il est impossible d'accompagner un chant arabe au piano parce que les intervalles de cet instrument ne sont pas suffisamment petits. M. CARRA DE VAUX suggère en terminant que l'on construise des orgues ou des pianos avec des quarts de ton, ce qui diminuerait de beaucoup les difficultés sans les supprimer entièrement. Mais ne vaut-il pas mieux laisser la musique d'orient se développer seule, dans sa propre direction ?

La traduction est suivie de notes savantes qu'il eut été tant plus commode de pouvoir lire au bas des pages : telles qu'elles sont, elles ne recevront pas l'attention qu'elles méritent. Je désire souligner l'une

d'elles. AL-FĀRĀBĪ commence son grand livre à la manière hellénistique :

« Tu as exprimé le désir de connaître l'art de la Musique tel que le concevaient les anciens. Tu m'as invité à écrire pour toi un livre traitant de ce sujet, facile à comprendre et à la portée de tous. J'ai tardé à exécuter tes ordres jusqu'à ce que j'aie lu attentivement les ouvrages des savants de l'Antiquité qui sont parvenus jusqu'à nous, ainsi que ceux de leurs successeurs et ceux de nos contemporains. J'avais espéré découvrir dans ces écrits ce que tu désirais savoir; j'aurais été ainsi dispensé de composer un ouvrage sur un sujet déjà traité... » (p. 1). « Dans les ouvrages que j'ai lus, j'ai trouvé que certaines parties de cet art étaient laissées de côté, que les dires de leurs auteurs manquaient de cohésion et de clarté, surtout dans ce qui touche à la théorie. On ne saurait imputer ces défauts à l'incapacité des auteurs anciens, ni laisser supposer qu'ils n'ont pas pu donner à cette science sa perfection. Ces savants étaient nombreux, pleins de talent; ils n'avaient d'autre idéal que le progrès de la Science. Ces gens d'une intelligence subtile se sont succédé; chacun étudiait les dires de ses prédécesseurs pour augmenter, à son tour, les connaissances qu'il avait reçues. » (p. 2).

L'idée de progrès est ici très clairement exprimée, et cela est fort remarquable, car cette idée est non seulement, comme le dit M. D'ERLANGER, une chose assez rare chez les auteurs anciens — elle y est absente. Je ne connais qu'une exception, très nette d'ailleurs, celle de SÉNÈQUE (3).

GEORGE SARTON.

Al-fihrist li Ibn al-Nadīm (Arabic edition of the *Fihrist* of IBN AL-NADĪM, with an Arabic introduction by one of the professors of the Egyptian University). 6+528+8 p. Al-muktabah al-tajāriyah al-kubrā, SHĀRI' MUḤAMMAD 'ALĪ, Cairo, 1348 (1929-30).

This new edition of the *Fihrist* does not deserve to be reviewed in *Isis* for its merits, but it is worthwhile to speak of it as an example and a warning to others. I bought it because it was announced as containing an additional part of the text *never published before* which was found in the library of the late TAIMŪR pāshā. (1) This statement repeated on the title page is false,—but not entirely so. The facts of the case are as follows.

GUSTAV FLÜGEL's edition of the *Fihrist* was posthumously published in Leipzig in 1871-72 (see my *Introduction*, I, 662). This edition was based upon a number of MSS. none of which is complete and perfect. An important part missing at the beginning of the fifth book and dealing with the origin of kalām and the biographies of its founders was edited

(3) Voir mon *Introduction*, I, 248; II, 484.

(1) This great Egyptian bibliophile and scholar has been many times quoted in *Isis* (e.g., 10, 249; 15, 405). See biography by A. SCHAADE : AḤMED TAIMŪR pāshā (1871-1930) und die arabische Renaissance (*OLZ*, 33, 854-59, 1930).

in 1890 by M. TH. HOUTSMA (*Vienna Oriental Journal* (2), vol. 4, 217-35, 1890). Either a reprint of HOUTSMA's paper or more probably a MS. copy of the additional text edited by him was inserted by TAIMŪR pāshā in his copy of the *Fihrist* (presumably FLÜGEL's edition of it).

The new editor, " one of the professors of the Egyptian University," got wind of that apparently when his own edition was almost completed, for otherwise why did he not publish the additional text at its proper place (p. 245)? He published it at the end of his volume with a new pagination, explaining that it was first printed in " Die Kunde des Morgenlandes " of 1889 and that he had found it in TAIMŪR's copy.

I did not bother to compare the Arabic text of both editions, as I shall continue to use FLÜGEL's edition and advise other scholars to do the same. The only possible advantage of the new edition, assuming it to be correct and complete, is that it combines HOUTSMA's text with FLÜGEL's, and that it is easily available; while FLÜGEL's two volumes have become rare and expensive. In any case the Cairene edition contains only the text itself, a short introduction, and an index: while FLÜGEL's edition includes an abundance of notes and elaborate indices, the whole of volume 2 being devoted to them (280 p., quarto, 1872). In short, from the scholar's point of view, the new edition may be considered non-existent.

The preface of the Cairene editor contains a brief account of the *Fihrist* and its author, showing how little the latter was known and how much his work was used by his successors. He discusses FLÜGEL's hypothesis that IBN AL-NADĪM was in Constantinople in 377 (=987-8) (3), and shows that the sentence " falaqaituhu bidār al-rūm warā' al-bī'ah " does not refer to Constantinople but to a Christian quarter and church of Baghdād. This conclusion had already been reached by Baron V. ROSEN in his Russian paper " Was the author of the Fihrist in Constantinople in 988? " (*Zapiski vost. otd. imp. russk. arch. obshch.*, vol. 4, for 1889: p. 401-4, 1890). However it is easier to read Arabic than Russian and I was interested in the discussion. The catholicos of Baghdād had sent a monk called AL-NAJRĀNĪ to China whence the latter returned after six years. IBN AL-NADĪM met this monk in Baghdād (not in Constantinople) in 987-88.

I take advantage of this review to mention two important publications relative to the *Fihrist*, which were omitted in my *Introduction* (vol. I, 662). SIEGMUND FRAENKEL: Zum Fihrist (*ZDMG*, 46, 741-43, 1892). JULIUS

(2) The title of this periodical in the copy used by me is actually in English as quoted.

(3) FLÜGEL, vol. 1, p. xiii, Arabic text, p. 349; vol. 2, 183.

LIPPERT : IBN AL-KŪFĪ, ein Vorgänger NADĪM's (*Wiener Zeitschrift für die Kunde des Morgenlandes*, 11, 147-55, 1897). It is worthwhile to add that some members of the faculty of the American University of Beirūt are now engaged in preparing an English translation of the *Fihrist*.

It is amusing to note that the Cairene edition of the *Fihrist* bears on its title page the mention "Printing rights reserved" (ḥuqūq al-ṭab' maḥfūẓah). Which rights? and how did they obtain them? I prefer not to use ugly words and therefore will leave that question unanswered.

GEORGE SARTON.

G. Tierie.—CORNELIS DREBBEL (1572-1633), H. J. PARIS, Amsterdam, 1932, (in English) VII+124 pages. Illustrated. Price, $ 1.20.

This handsomely printed book is a biography of a great inventor who failed to be a great scientist because he was not financially independent (like ROBERT BOYLE and CHRISTIAAN HUYGENS). Like other inventors and artists of his day he was largely dependent upon the favor of princes, and was obliged to do everything in his power to maintain his position at court. He was forced to wrap his inventions and demonstrations in secrecy and to say very little about his theories.

At the age of twenty-five DREBBEL was already an engraver of unusual merit, having been trained in the art by HENDRICK GOLTZIUS, his wife's brother by whom he was also instructed in alchemy. At about the same time he began his career in applied science by studying the effects of pressure and temperature on the volume of air. In 1598 he took out a patent for an automatic pump and for a clock which did not need winding, both of them being operated by the variations in the volume of a confined quantity of air. In 1600 he built a fountain at Middelburg which operated on the same principle; in 1605 while in England he built his *perpetuum mobile* for JAMES I at Eltham; and in 1610 while in Bohemia he built another fountain for RUDOLF II at Prague. The *perpetuum mobile* in its simplest form consisted of a glass bulb attached to a spiral of glass tubing in which there was a small quantity of liquid. The gas in the bulb, expanding by day and contracting by night, produced an ebb and flow of the liquid in the spiral tube. There is some evidence which indicates that DREBBEL perhaps filled his bulb with oxygen gas procured by the heating of saltpeter. On the same principle he constructed self-registering ovens and incubators for the hatching of chicks. He was expert in the manufacture, blowing, and grinding of glass, and invented an ingenious machine for the grinding of lenses. His *camera obscura* attracted much attention from artists and savants. He is credited with the invention of the microscope with two convex lenses.

DREBBEL won much fame with his submarine or diving-boat in which he went underwater down the Thames for a considerable distance in the presence of JAMES I and his court. The boat had no bottom and was constructed on the principle of the diving bell. He measured his depth under water by means of a mercury barometer, steered by means of a compass, and renewed the goodness of the confined air in the boat by the use of a gas made from saltpeter, ' broken up by the power of fire and so changed in the nature of air.' He believed that air is made up of two component parts, the *quintessence* (oxygen), fit for respiration, and the *carcass* (nitrogen), unfit for respiration.

Soon after his arrival at the English court, DREBBEL helped to arrange a fireworks display. Later he devoted himself to the preparation of fulminating gold, a substance which he possibly discovered, which he used in 1628 before La Rochelle as a detonator in his petards and torpedoes. He was the first to use tin salt as a mordant in dyeing with cochineal. After his dismissal from the British Navy in 1629, he lived in poverty in London—and died in the Parish of Trinity in 1633.

His principal writings are " A short treatise on the Nature of the Elements and how they bring about wind, rain, lightning, thunder, and why they are useful," first published in Dutch in 1604, and " De Quinta Essentia," first published in Latin in 1621.

Dr. TIERIE's book is supplied with useful Summary and Notes, and with an Index of Names. T. L. D.

Mary Sturt. *Francis Bacon, a biography.* XVI+246 p., 12 pl. New York, WILLIAM MORROW & Co. 1932. Price $ 3.50.

When FRANCIS BACON left Cambridge he wrote that " all the philosophy of nature which is now received is either the philosophy of the Grecians or that other of the Alchemists. That of the Grecians hath the foundations in words, in ostentation, in confutation, in sects, in schools, in disputations. That of the Alchemists hath the foundation in imposture, in auricular traditions and obscurity."

The author devotes two chapters to BACON's scientific work, one to *The Advancement of Learning* and one to *The Novum Organum*. The *Sylva Sylvarum*, which reflects BACON's life-long devotion to gardens and gardening, she, perhaps wisely, leaves unnoticed. It is rather a random sampling and a rambling treatment of incoherent natural phenomena in which little, if any, realization of the inductive method can be detected. The *Advancement* was written during nine months in 1605-1606 while Parliament was not in session through fear of the plague. BACON was a modern in that his mind adjusted quickly to diverse activities. For this reason his accomplishments were both prodigious and varied.

The introduction to this work reveals BACON's ideas on science. He was no scientist, made no discoveries, lived and died an amateur, yet his ideas are so in accord with those of today that any international scientific society might turn to him for a motto. He lived when authotity reigned, when experiment was not practiced, and cooperation in institutions and of international character was unknown. But he foresaw their need and envisioned a College for Inventors with equipment (' inginary "), library, with provision for experimentation, travel, and for " intelligence mutual with the universities of Europe." This idea appeared in its grandest form in his *New Atlantis*. He pleaded for learning freed from professional aims and for one left free to the arts and sciences at large. He deplored the scholars " who sell books and build furnaces, forsaking Minerva for the service of Vulcan."

The *Novum Organum* (1629) was designed, as BACON wrote to the King, " to enlarge the bounds of Reason and endow man's estate with new value." His elaborate system of induction, set out in axioms of extreme ingenuity and adorned with phrases, was barren of result. His method was premature. He attempted to set out the course which science was to follow when experimental science was as yet unborn. He hoped to turn men to nature and away from their spider-web subtleties.

BACON wished to control nature for man's good, therefore man must understand nature, and to understand nature he must first observe her and collect his observations. Then he must form hypotheses as to the cause of the phenomena he observes, check these hypotheses against the observations, and so arrive at truth. He understated the significance of hypothesis, as he was somewhat fearful of its being overdone. Though he emphasized experiment, he rarely practiced it, except in the instances which cost him his life. Miss STURT attempts throughout to remove the stigma which MACAULAY cast upon BACON's character, though her defense against his accusers of accepting bribes is rather lame. The book suffers from lack of sufficient dates or chronology. Two portraits of BACON are included, the one as Chancellor, from the National portrait Gallery, and an interesting one of him as a boy without source or date.

December 13, 1932. C. A. K.

Hermann von Helmholtz; Max Dohrn; Julius Schiff.—*Goethes naturwissenschaftliches Denken und Wirken.* Drei Aufsätze herausgegeben von der Schriftleitung der Zeitschrift *Die Naturwissenschaften.* 99 p., pl. Berlin, JULIUS SPRINGER, 1932. (RM. 2.60; bound, RM. 3.60).

Eine besonders schöne Gabe zum GOETHE-Gedächtnis Jahr für alle naturwissenschaftlich Interessierten sind drei in einem Bändchen von

der Schriftleitung der "Naturwissenschaften" herausgegebene Vorträge, die GOETHES Beziehungen zu den verschiedenen Zweigen der Naturwissenschaft darstellen.

Zunächst finden wir eine Rede von HERMANN VON HELMHOLTZ über „GOETHES Vorahnungen kommender naturwissenschaftlicher Ideen", die er im Jahre 1892 anlässlich einer Versammlung der GOETHE-Gesellschaft zu Weimar hielt.

HELMHOLTZ, der damals auf der Höhe seines wissenschaftlichen Lebens stand und den grossen Aufschwung der Naturwissenschaft in der zweiten Hälfte des vergangenen Jahrhunderts als einer der grössten Vorkämpfer miterlebte, sucht dichterisches Schaffen und die Bestrebungen in die Geheimnisse der Naturgesetze einzudringen, in ihrer Grundform darzustellen. Obwohl die Wege, die vom schaffenden Künstler und vom Forscher eingeschlagen werden bei der Ausübung ihrer Tätigkeit verschieden sind, so sind die primären Grundbedingungen für beide die gleichen, nämlich die der Wahrnehmung. Für den Künstler kommt hinzu das Gedächtnis, das ihm erlaubt, das Wahrgenommene festzuhalten und bei der Ausarbeitung wiederzugeben, für den Forscher ist die Ahnung neuer Gesetzmässigkeiten von Bedeutung. GOETHE vereinigt in sich Künstler- und Forschernatur und so finden wir in seinen Dichtungen und naturwissenschaftlichen Dokumenten Vieles was Dichter und Forscher in seiner Persönlichkeit vereinigen.

Unter den naturwissenschaftlichen Fragen, wobei das Suchen nach Urphaenomenen eine der Hauptideen bei GOETHES Forschungen darstellt, spielt die Analyse des Urphaenomens der Erregung eine bedeutende Rolle. Auf physikalischer Seite finden wir dies am klarsten in der "Farbenlehre". Unglücklicherweise scheiterten diese Studien "wesentlich daran, dass er mit verhältnismässig unvollkommenen Apparaten, die er in Händen hatte, die entscheidenden Tatsachen nicht hat beobachten können". Jedoch liegt in ihnen ein wesentlicher Kern für die Weiterentwicklung von GOETHES eigenen Forschungen und die der Naturwissenschaft überhaupt. Aus der Verworrenheit früherer Jahrhunderte bleibt die Suche nach dem Urphaenomen, von dem GOETHE glaubt, dass es in jedem Zweig der Physik zu finden sei. Man wendet sich gegen mystisch-spekulatives Denken "und so hat die Physik Wege eingeschlagen, auf die GOETHE sie führen wollte". Andere Forscher seiner Zeit, die mit der mathematischen Seite der Physik vertrauter waren, strebten wie er alle dem einen Ziel zu "die in der Natur vor sich gehenden Bewegungen vollständig und auf die einfachste Weise zu erklären". Obwohl dieser Satz erst viel später von KIRCHHOFF ausgesprochen wurde, so spricht aus ihm wieder GOETHES Gedanke über das Suchen nach den Urphaenomenen. Wie in der Farbenlehre so finden wir in anderen

Schriften weitere grundlegende Fragen in undeutlich umrissener Form angedeutet, so finden wir das Gesetz der Erhaltung der Energie, das Gesetz der spezifischen Sinnesenergie und manches andere in Dichterworte gekleidet. Von besonderem Interesse sind ferner HELMHOLTZ's Worte über GOETHES Beziehungen zur Philosophie. GOETHE selbst sagt beim Studium von KANTS "Kritik der reinen Vernunft" : "Der Eingang war es der mir gefiel, ins Labyrinth konnte ich mich nicht wagen ; bald hinderte mich die Dichtungsgabe, bald der Menschenverstand, und ich fühlte mich nirgends gebessert". Hier zeigen sich wieder die Dichternatur, die von philosophischem Denken frei sein möchte, und die Forschernatur, die sich an die Tatsachen hält. HELMHOLTZ fasst seine Gedanken über GOETHES Bedeutung für die Naturwissenschaft folgendermassen zusammen : "Wo es sich um Aufgaben handelt, die durch in Anschauungsbildern sich ergehenden dichterischen Divinationen gelöst werden können, hat sich der Dichter der höchsten Leistungen fähig gezeigt, wo nur die bewusst durchgeführte inductive Methode hätte helfen können, ist er gescheitert. Aber wiederum, wo es sich um die höchsten Fragen über das Verhältnis der Vernunft zur Wirklichkeit handelt, schützt ihn sein gesundes Festhalten an der Wirklichkeit vor Irrgängen und leitet ihn sicher zu Einsichten, die bis an die Grenzen menschlicher Vernunft reichen."

Im zweiten Teil des Buches wird in einem Vortrag von MAX DOHRN : „GOETHE über seine naturwissenschaftliche Denk- und Arbeitsweise", durch zahlreiche aus GOETHES Schriften zusammengetragene Stellen, die Entwicklung seines Interesses an naturwissenschaftlichen Fragen und deren Bedeutung dargestellt.

Schon während GOETHES Studium ist das Interesse an naturwissenschaftlichen Fragen bedeutend. Durch seine Berufung und die Übernahme der Staatsgeschäfte am Hofe von Weimar wird sein Verhältnis zu praktischen naturwissenschaftlichen Fragen ständig grösser, und da er mit der Verwaltung der Universität Jena betraut ist, findet er durch den Kontakt mit den Vertretern verschiedener Wissensgebiete reichliche Anregung zu seinen Studien. Die Botanik, die damals durch den Einfluss LINNÉS in wesentlich andere Bahnen geleitet wurde, interessiert ihn besonders. So veranlasst ihn die Suche nach einer Urform bei der Pflanze zu seinen Studien der Pflanzenmorphologie, der Pflanzenmetamorphose, wobei Fragen der Hormonlehre, Oekologie und Reizphysiologie auftauchen. Die Leitung von Bergbaugeschäften macht ihn mit geologischen Problemen vertraut. Seine Studien über Vulkane und Tiefengesteine und seine Schlüsse auf die Ausdehnung von Glacialperioden aus erratischen Blöcken, sowie Kartenaufnehmen sind für die Weiterentwicklung dieser Wissenschaft von Bedeutung. Dazu kommen

Interesse an der Meteorologie, die zu Wolkenbeobachtungen Veranlassung gaben und ihn dank der Kraft seines Amtes in die Lage setzten, meteorologische Stationen errichten zu lassen. In gleicher Weise interessierte er sich für die Tierwelt. Von anatomischen Studien und Unterricht, den er selbst erteilte, ausgehend schrieb er seine vergleichend anatomischen Werke, die bis zur Wirbeltheorie des Schädels führten. In der Tierwelt sucht er wie bei allen seinen Studien nach dem Ursprünglichen, dem Typus. An vielen Stellen finden sich Anspielungen auf die morphologische und physiologische Harmonie im Innern eines Organismus, und wie das Beziehungsverhältnis für den Tierkörper an sich ihm klar erscheint, so sucht er es zwischen Individuum und Aussenwelt. GOETHE sieht hierin ein Mittel die Entwicklung bestehender Formen aus Grundformen abzuleiten und Varationen teils durch „die dem Typus innewohnende Natur", teils durch Einflüsse des Milieus zu erklären. Damit sind die Grundgedanken der Descendenztheorie ausgesprochen, die aber erst von DARWIN durch Mitinbetrachtziehen der Selectionsidee in ihrer endgültigen Form dargestellt worden ist. Von besonderem Interesse sind alle die angeführten Stellen, in welchen GOETHE sich darüber ausspricht, dass die damalige wissenschaftliche Welt seine Untersuchungen ablehnte, und wie sehr ihn dies betrübte, da er seine wissenschaftlichen Leistungen für viel wesentlicher hielt, als sein dichterisches Schaffen.

Im dritten Teil des Buches hat JULIUS SCHIFF "Naturwissenschaftliche Gleichnisse in GOETHES Dichtungen, Briefen und literarischen Schriften" zusammengetragen. Aus der Welt der Erscheinungen, die GOETHE bei seinen naturwissenschaftlichen Studien beschäftigte, sind zahlreiche Gleichnisse herausgegriffen und in seine dichterischen Schriften und Briefe eingewoben. Die Mannigfaltigkeit ist enorm. SCHIFF hat sie nach den verschiedenen Zweigen der Naturwissenschaften geordnet. Beginnend mit der Physik, finden wir Stellen angeführt, wo GOETHE die Eigenschaften der Materie, Gesetze der Mechanik, Akustik und Wärmelehre, der Optik und Electrizität zu Gleichnissen heranzieht. In gleicher Weise finden wir Beispiele, die von der Chemie genommen sind. Ihnen folgen solche aus Astronomie, Mathematik und Meteorologie. Dann schliessen sich solche aus der Geologie und Mineralogie, Biologie und Medizin an. Es würde zu weit führen zu versuchen Einzelheiten dieser ausserordentlich interessanten Zusammenstellung wiederzugeben. Beim Studium von GOETHES Schriften wird man immer auf naturwissenschaftliche Gleichnisse stossen. Ihre grosse Zahl und Mannigfaltigkeit kann einem jedoch dabei leicht entgehen, und so gibt SCHIFFS Beitrag zu dem Buche einen besonders schönen und wertvollen Einblick in den Reichtum der Gleichnisse, die GOETHE in seine Dichtungen eingewoben hat.

ERNST WOLF.

Henry Fairfield Osborn (with the cooperation of HELEN ANN WARREN). —*Cope: Master Naturalist.* The life and Letters of EDWARD DRINKER COPE, With a Bibliography of His Writings Classified by Subject. A Study of the Pioneer and Foundation Periods of Vertebrate Palaeontology in America. Princeton University Press, Princeton. XVI+740 p., 30 figs. in text, maps on end papers and key to COPE's travels on fly leaf, 1931. (Price $ 5.00).

No one among the palaeontologists of the world is so well prepared to write the life of this remarkable personality and preeminent contributor to the evolution of vertebrate life as his friend and fellow palaeontologist, Professor HENRY FAIRFIELD OSBORN. The biographer's work has been a labor of love initiated before COPE's death and continued intermittently for thirty-three years. Since so much of OSBORN's own research has also been in the field of vertebrate palaeontology, he is in a position to fully understand the extent and significance of COPE's contributions and to appreciate the difficulties and rivalries he met in accomplishing them. He also had the assistance of the COPE family and access to the brilliant and voluminous correspondence of COPE now deposited in the American Museum of Natural History in New York.

The book opens with a brief chapter on the pioneers of palaeontology in North America, which includes a list of 98 genera and species of fossil vertebrates described prior to 1847 and a summary of the work of the fathers of American vertebrate palaeontology, LEIDY, COPE, and MARSH, who between 1847 and 1897 described no less than 2193 new genera and species. Western exploration early revealed the extraordinary richness of the Plains of North America in the fossil remains of vertebrates, especially of reptiles and mammals. The contributions from these sources, notably to the evolution of the horse, have been of inestimable value in rounding out the historical evidence for the evolution of the vertebrates. COPE's part in this program was a basic one. He originated practically all that is known of the rhinoceroses, horses, mastodons, camels, and carnivores of the Upper Miocene. His personal activity in field collecting enabled him to enter the field as a pioneer in stratigraphic correlation, both in North America and elsewhere, especially in the case of the fresh-water Tertiaries. His explorations in Texas confirmed the occurrence of the Permian in the United States.

COPE was by far the greatest herpetologist of America. He wrote the first extensive texts in this field and utilized anatomical characters as a basis for phylogenetic interpretations, as well as for classification. Specifically, he established the primitive nature of the arciferal pectoral girdle and the relative unimportance of the presence or absence of teeth in the classification of the Anura. He based his classification of lizards,

afterwards generally followed, on skeletal as well as superficial structures.

His contributions to ichthyology include accounts of Eocene and Cretaceous fishes ; a major classification largely erroneous, except for the detection of many new orders still held as valid ; and the breaking up of the unnatural orders of Ganoids and Apodes. His work in this field was inferior to that of GILL.

The most numerous and most voluminous writings of COPE deal with the Mammalia. CUVIER's extensive work in this field had been dominated by the non-evolutionary point of view which had affected both interpretation and classification. The work of OWEN, DE BLAINVILLE, and LEIDY had advanced the subject but little beyond the recognition of the Law of Brain Growth and the Classification of Hoofed Animals by Foot Structure. With the coming of Darwinism the evolutionary principle was invoked. HUXLEY overthrew CUVIER's Law of Correlation ; KOWALEWSKY announced the Law of Adaptation of Foot Structure in Ungulates by Reduction of Digits ; and RYDER attacked the problems of the mechanical evolution of feet and teeth from the Lamarckian standpoint.

COPE entered this field with enthusiasm and added to the generalization of his predecessors four principles. The first was that the ancestors of the hoofed animals had hillock-like or bunodont teeth, verified by later discoveries. This led to a reclassification of the Ungulata centering about his second principle, his Law of Taxeopody, that the primitive feet of hoofed animals were serially plantigrade. Freed from certain errors, his generalization affords a basis for the phylogenetic arrangement of the hoofed orders now widely used. The third and perhaps greatest generalization was based on evidence derived from the Puerco fauna. This was the Law of Tritubercaly ; that all types of molar teeth in mammals originate in modifications of the tritubercular form. This law promises ultimately to give the key to the entire phylogeny of the Mammalia. His fourth principle was that the Hoofed Orders arise from the Clawed Types of Creodonta and Insectivora.

COPE was quick to perceive the significant facts of structure, especially those diagnostic of relationship and origin. Thus representatives of the order Creodonta had been discovered and studied for years in France but without coordination. It took the eagle eye of COPE to recognize the structural characters which unified and characterized this very fundamental order of mammals.

In the field of evolutionary theory COPE was always a confirmed Lamarckian, not only in principle, but often in form as well. He was early attracted to the failure of DARWIN's theory to explain the origin of variations. In his Walker Prize Essay of 1872 on *The Method of*

Creation of Organic Forms he distinguished between the originative and directive forces in evolution and concludes that "natural selection includes no actively progressive principle whatever." His views are elaborated in *The Origin of the Fittest* (1886) and *The Primary Facts of Evolution* (1896).

The rise of Neo-Darwinism led by WEISMANN stimulated his papers on *The Mechanical Causes for the Development of the Hard Parts of the Mammalia* (1888) and his address before the American Society of Naturalists on *The Inheritance of Acquired Characters* (1896). His theoretical work was an expansion of LAMARCK's doctrines couched in a new terminology of Archaesthetism, Primary Sensation, Bathmism or Growth Force, and Kinetogenesis or the energetic origins of all *adaptations* and hard parts of mammals. He was a defender of orthogenesis, the inheritance of acquired characters, and of the direct effect of the environment.

COPE favored the Darwinian theory of the origin of man from the anthropoid apes and agreed with HAECKEL as to the lemuroid origin of the Anthropomorpha. His view as to the evolution of mammals rested on the concept of a succession of stages rather than on the profoundly branching system of OSBORN.

This biography is enriched with many classified quotations from the virile, sketchy, humorous, and illuminating correspondence of COPE on a wide variety of subjects, anatomical, osteological, palaeontological, stratigraphical, controversial, evolutionary, social, political, and religious. There are numerous portraits and a classified bibliography of 148 pages. There are also reproductions of sketches from his note books.

No biography of COPE would be complete without recognition of his relations to *The American Naturalist*, a relation not specifically emphasized in OSBORN's life of this great naturalist. He was one of the early contributors to its pages, took a large part in financing it, and for many years edited it, wrote editorials and reviews for it, and was one of its most regular contributors. This service in maintaining this pioneer general biological periodical was one of his greatest contributions to American natural science. The shift in emphasis in this journal in recent years from natural history and palaeontology to experimental biology and genetics is indicative not only of the diversification of the natural sciences, but also of changes of method of emphasis, and of point of view with reference to the fundamental factors in evolution since COPE theorized about the inheritance of the effects of use and disuse.

December 1, 1932. C. A. KOFOID

Clara G. Stillman.—*Samuel Butler, a Mid-Victorian Modern.* The Viking Press, New York, XI+319 p., 6 pls., 1932. (Price $ 3.75).

Mrs. STILLMAN's biography of SAMUEL BUTLER (1837-1902) is a sympathetic presentation and critical psychological analysis of the picturesque personality of this rebel and intellectual outcast of the Victorian period. BUTLER was repressed as boy by strict parental discipline. The defense mechanism thus developed made him the champion of change, a persecutor of shams, and a relentless pursuer of the hypocrite, real or imagined. But throughout his life he was an amateur in the Great Period of amateurism, seeking in turn to be a sheep rancher in New Zealand, an artist (of insignificant mediocrity), a linguist, a musician and composer, a philosopher, an anticipator of modern behaviorism in psychology, a Homeric critic, a novelist, a revolutionary sociologist, and, what most concerns us here, a critic and a creator of evolutionary theory. In all of these fields he was a miserable failure—in his own day. He perforce had to publish his own books, and when published they were ignored, or, if noticed, generally ridiculed, or at the most drew forth only casual and unsignificant comment.

But he anticipated the temper, and himself lived in the thought, of the present century and especially of post-bellum days. His satire, keen scent for the vulnerable, and his ceaseless and ever-shifting search for reality, have won for him the epithet " that keen fox " and a widely spreading recognition of the prophetic aspects of his incursions into varied fields of criticism, literature, philosophy, and science.

Mrs. STILLMAN's sympathetic and rather full and critical analysis of his character and reactions to ideas, situations, and personalities is even more helpful in our understanding of his relations with DARWIN than the exhaustive but labored account by his personal friend, FESTING JONES, or than the scientific presentations of HARTOG or Sir FRANCIS DARWIN.

The Butler-Darwin controversy was a one-sided affair, for DARWIN, on the advice of HUXLEY and against the wishes of FRANCIS DARWIN and his brothers, did not send or publish a letter which he had drafted which would have explained the contradiction which had aroused BUTLER's ire. This failure on the part of DARWIN to remove the occasion for the charge of unfairness was the psychological basis for BUTLER's prolonged hostile attitude towards DARWIN, for whom in earlier years he had expressed his ardent admiration.

The history of this unfortunate affair was related in 1911 in a small pamphlet, written by HENRY FESTING JONES after conference with Sir FRANCIS DARWIN, entitled *Charles Darwin and Samuel Butler, a Step towards Reconciliation.* But even this fails to state certain basic

environing factors which serve in part to clarify, if not to extenuate, DARWIN's action, or more specifically, failure to act, to which BUTLER took offense. One of these factors was the then existing state of scientific ethics in matters of priority and acknowledgment of credit. ENGELMANN's *Bibliotheca Historico-Naturalis* 1700-1846 was published only 14 years before the *Origin of Species* (1859). It was not the general practice to cite authorities and to acknowledge priorities of ideas, especially in treatises of a general sort, such as the *Origin*. DARWIN was, however, exceptionally punctilious in acknowledging his sources of information on particular matters of fact, and after BUTLER's assaults, in the third and later editions of the *Origin*, he introduced and progressively improved his *Brief but Imperfect Sketch of* earlier evolutionists.

A second factor, subtile but potent, was the British antagonism to the "dangerous speculations of the French Revolutionary School" which grew in violence up to and after Waterloo, but was less in evidence when ERASMUS DARWIN wrote his *Zoonomia*. BUFFON and LAMARCK, and ERASMUS DARWIN with them, fell into disrepute. Editions of BUFFON, once popular in Great Britain, ceased to be published there, and LAMARCK's theories attracted no sympathetic attention or even translation. Catastrophism was still popular at the Universities even after LYELL's uniformitarianism had arrived, and transformism has not today become Anglicized. DARWIN was probably ignorant of the precise contents of the works of these his most brilliant predecessors. In any event he was more intent upon the putting over of his idea of natural selection than concerned with reviewing from unpopular sources past efforts to establish the other evolutionary factors which he had incorporated in the *Origin*.

This omission aroused the combative defense mechanism of SAMUEL BUTLER and was the central theme of his *Evolution, Old and New* (1879). It reappears in his later works *Luck, or Cunning* (1887), and especially in *Unconscious Memory* (1880) in which he slyly uses DARWIN's unacknowledged appropriation of his grandfather's ideas to point his moral. He returned to the attack in his series of articles in the *Universal Review* on *The Deadlock in Darwinism*

But BULTER had a deeper and more penetrating, if less exasperating, disagreement with DARWIN, and one which our biographer is inclined to think DARWIN sensed, and, in declining years of poor health, not only dismissed, but actually shrank from. He was not unaware of the keen critical power, sharp nose, and sharper pen of his antagonist. He had ignored his first assault, and continued to do so, to the rage and chagrin of BUTLER, but did not wish to incite further controversy. It is, however, more probable that BUTLER's metaphysical theory of the origin of variations through the internal factor of unconscious memory

lay so far beyond observation and record that DARWIN's matter-of-fact mind refused to evaluate it.

BUTLER's first contribution to evolutionary theory was made in his *Life and Habit* (1878). He was challenged by DARWIN's failure to explain the origin of variations and set about finding one. This brought him to the concept of internal factors, the aim, purpose, functioning, or, as he later phrased it, the unconscious memory, of the organism. It also was the source of his interest in Lamarckian evolution and links him to the school of vitalism. It is, however, a modified form of that doctrine in which the unconscious effort or will of the organism itself is substituted for a General Intelligence. The continuity and progress of evolution which the orthodox vitalist seeks to explain is provided for in BUTLER's philosophy of evolution by his concept of the continuity of parent and offspring, foreshadowing WEISMANN's Continuity of the Germ Plasm, and by the extension of this continuity to include all life in the concept of a general organism. The physical basis being thus postulated, it is easy to take the next logical step and postulate new needs giving rise to new efforts and the transfer of the adaptive accomplishments by the mechanism of the continuum of the living substance to the next generation. BUTLER first intimated this philosophy in two letters to the New Zealand Press shortly after the publication of the *Origin*, entitled *Darwin among the Machines* and *Lucubratio Ebria*. In these he clearly brought out the distinction between natural selection and his own emerging hypothesis. Critics of DARWIN seized upon these ideas when they were incorporated in *Erehwon*, but BUTLER in letters to DARWIN disclaimed intent of criticism. His theory came to fullest expression in his *Unconscious Memory* (1880), in which he compared it with HERING's *Memory as a Function of Organized Matter*, acknowledging its identity thereto, and with VON HARTMANN's *The Unconscious in Instinct* in his *Philosophy of the Unconscious*. In VON HARTMANN he finds only mystical assumptions of the clairvoyant powers of the unconscious, but no reasoned theory.

The strife between mechanism and vitalism has not ended. External factors lend themselves to experiment, and the internal ones are still elusive, though the gene hypothesis and cytology unite to give a substantial basis for a physical continuum which BUTLER envisaged. But this is a continuum of particulate rather than organismal configuration. Changes in its patterns may be stimulated by external factors, but they are not specific to the stimulus, are rarely adaptive, and afford more satisfaction to the Darwinian than to the Lamarckian-Butlerian school of evolutionists. But the division still continues. What a pity BUTLER's pen has ceased to write!

December 1, 1932. C. A. KOFOID.

Houston Peterson.—*Huxley, Prophet of Science.* LONGMANS, GREEN and Co., London, XIV+338 p., 4 pls. 1932. (Pr. $ 3.50).

This is a sincere effort to appraise the character, scientific work, and influence of HUXLEY. The author is known for his *Havelock Ellis, Philosopher of Love* and for his *Melody of Chaos*, centered around the poetry of CONRAD AIKEN. He had access to HUXLEY's note books and drawings in the library of the Royal College of Science at South Kensington and publishes in an appendix the last contribution of HUXLEY, which never got beyond galley proof, entitled *Mr. Balfour's Attack on Agnosticism*. The author traces the influences which shaped the career of HUXLEY : his boyhood discussions with MAY, the village merchant, over *Divine Government* ; the influence of Sir WM. HAMILTON's *Philosophy of the Unconditioned* ; his fondness for CARLYLE's *Sartor Resartus* ; and through CARLYLE the interest aroused in the writings of GOETHE ; the model set before him by the logical and lucid lectures of his professor of anatomy, THOMAS WHARTON JONES ; the dislike of sermons created in him as a lad by enforced attendance at church ; the scientific training acquired in laboratory practice in his four years' voyage on the *Rattlesnake*, with abundant time for meditation ; his early introduction into practical politics in his relations with the British Admiralty ; his facility in making friends and acquaintances in influential circles ; and his growing skill in argument and dialectics.

Our biographer has a penchant for the psychological, and attributes HUXLEY's boyhood and life-long recurrent headaches and chronic dyspepsia to his early experiences as a lad at an autopsy, and a resulting fear neurosis and his occasional freedom from attacks when fighting bishops, bad scientists, and believers generally, to the relaxation of this deadly tendency. Much more probable, as well as scientific, is the explanation by Dr. GOULD in his *Biographic Clinics*, that the ill health of HUXLEY, as also that of DARWIN, was due to eyestrain.

The author skillfully portrays HUXLEY's growing distrust of Sir RICHARD OWEN and his final falling out with him, which arose over OWEN's errors in magnifying and distorting the differences between the brain of man and of the anthropoid apes. This controversy culminated in HUXLEY's first distinctive work on evolution, *Man's Place in Nature*.

HUXLEY's relation to DARWIN is adequately treated by citations from correspondence indicating how DARWIN refrained from revealing his theory of natural selection to HUXLEY until it was in print in *The Origin of Species*, and how concerned, even fearful, DARWIN was as to HUXLEY's opinion of it. HUXLEY was always somewhat of a believer in the fixity and definite delimitations of species and other categories of classification and devoted one of his major addresses to the instances of the persistence

of certain genera throughout long reaches of geologic time. In HUXLEY's reply to LYELL's objections to DARWIN's theory on the ground that there were insufficient signs of gradual change in organisms, he elaborated his opinion that transmutation of species might take place without transition, using analogies from chemical changes. In this he was more in agreement with MENDEL, BATESON, and DE VRIES than was DARWIN, though geneticists are now dealing with differences of the order of magnitude quite Darwinian in proportion, though still involving phenomena of discontinuity.

The large part which HUXLEY played in the drama of the coming of organic evolution, as a factor influencing many fields of human thought, is related in detail with regard to his public addresses and polemic writings and controversies, notably those with Bishop WILBERFORCE, Sir RICHARD OWEN, CHARLES KINGSLEY, W. G. WARD, GLADSTONE, Canon LIDDON, and BALFOUR. He lectured extensively to sympathetic audiences of laboring men, but apparently never became militant in their behalf.

Biological science owes much to HUXLEY for his influence on teaching. He introduced and fostered the laboratory method, supplementing didactic presentation from the lecture platform. HUXLEY and MARTIN's *Practical Instruction in Elementary Biology* set the pattern for laboratory courses for more than a generation and is responsible for the type method in the presentation of morphology. There is a bibliography of HUXLEY's major writings, of books about him, and of works descriptive of the interpretation of the thought of the Victorian period, in the shaping of which he played no small part.

November 30, 1932. C. A. KOFOID.

Clarence Ayres.—*Huxley*. 254 p., frontispiece. New York, NORTON. 1932. (Price $ 3.00).

Mr. CLARENCE AYRES' life of THOMAS H. HUXLEY is written in the modern style of biography, playing up all that is dramatic and tragic in the life of this great biologist, and displaying all phases of HUXLEY's character which support the thesis that he was "DARWIN's bulldog," if not indeed his co-equal. But Mr. AYRES goes much farther, even so far as to assert and defend the claim that HUXLEY, not DARWIN, was in reality the author of the theory of evolution as generally understood today. DARWIN was interested in the problem of the origin of species, HUXLEY in the descent of man from the anthropoid type. HUXLEY's break with OWEN occurred when he spiritedly attacked the inaccuracies of the latter's paper at the British Association in 1860 on the differences between the brain of man and that of the anthropoid apes. This paper claimed, among other differences, that the anthropoid brain contained

no hippocampus minor, and HUXLEY's contention and corrections gave prominence to and strongly determined the trend of much of his later work. To OWEN's claim HUXLEY gave impromptu a direct and unqualified contradiction and later (1861) published his crushing review of OWEN's paper. This at once brought forth the epithet "gorilla book" for DARWIN's *Origin of Species*, although the human species is mentioned but once in the entire work, and that on the penultimate page.

DARWIN is charged by Mr. AYRES with being timid, even psychopathically so, and that he feared the popular repercussion against his pet theory of origin by natural selection if man's descent from the ape should be dragged into the foreground. He notes that DARWIN's *Descent of Man*, published eight years (1871) after HUXLEY's *Man's Place in Nature*, was written mainly in support of DARWIN's theory of sexual selection. In the introduction to this work DARWIN cites his indebtedness to HAECKEL's *Natürliche Schöpfungsgeschichte* (1868, ed. 2, 1870), but lumps HUXLEY's unnamed work with those of others, although he refers only eleven times to HAECKEL's book and fifteen times to HUXLEY's. The inference HUXLEY's biographer seems to wish his readers to make is that DARWIN minimized what HUXLEY magnified, namely the application of the Darwinian theory to the origin of man himself.

No one will deny the services to the cause of organic evolution of the great apostle of Darwinism, but an inspection of the *Descent of Man* and the *Expression of the Emotions* rather indicates that DARWIN was quite unsuccessful in hiding his interest in the intimate relationship of anthropoids and men. Our biographer has stuffed his timid DARWIN full of straw as a foil for the enhancement of the career of the more aggressive and militant scientist who was primarily interested in the application of the doctrine of descent to the vertebrates rather than in the dynamics of the process. Obviously, present-day evolutionists are still as a body more interested in this dynamic field than in its limited application to one species. It is only a militant remnant of the defeated enemy who still drag out the gorilla. HUXLEY's own estimate of his part in the establishment of evolution is happily given in his address of acceptance of the DARWIN medal, a signal honor overlooked by our biographer. He disclaimed equality with WALLACE and HOOKER, the first to whom the medal had been awarded, and claimed only that he was "convinced, as firmly as I have ever been convinced of anything in my life, that *The Origin of Species* was a ship laden with a cargo of rich value, and which, if she were permitted to pursue her course, would reach a veritable scientific Golconda, and I thought it my duty, however naturally averse I might be to fighting, to bid those who

would disturb her beneficent operations to keep on board their own ship."

December 1, 1932. C. A. K.

J. Harland Paul.—*The last cruise of the Carnegie*. With a foreword by JOHN A. FLEMING. XVII+331 p., 198 ills. Baltimore, WILLIAMS & WILKINS, 1932. ($ 3.00).

The Carnegie Institution of Washington is divided into departments, one of the most important of which is the Department of Terrestrial Magnetism, founded in 1904 by the late Dr. LOUIS AGRICOLA BAUER (1865-1932). The purpose of the Department is " to investigate such problems of world-wide interest as relate to the magnetic and electric condition of the Earth and its atmosphere, not specifically the subject of inquiry of any one country, but of international concern and benefit." In order to study magnetic and electric conditions at sea, and to make possible the improvement of marine charts, a brigantine was chartered in 1905 and named the *Galilee*. She made three cruises, but was very uncomfortable and inadequate.

" Since she was not free from magnetic materials, it was necessary to ' swing ship ' for deviation-errors as often as circumstances permitted. These errors occasioned delay in reporting results, and made the computations of final values most laborious. It was apparent that a non-magnetic vessel with observatory domes would be able to do the work far more efficiently. It was in answer to those needs that the *Carnegie* was built in 1909. The experience with the *Galilee* had been invaluable; old instruments were adapted to marine use, new ones were invented, and methods were compared with the aim of finding those which gave the greatest accuracy under ever changing conditions at sea.

The *Carnegie* made six cruises between 1909 and 1921"... " During these she sailed more than a quarter of a million nautical miles, making some of the longest voyages in history, and traversing all waters between 80° north and 60° south. She had met ice and fogs around Spitzbergen and the South Orkneys, typhoons off Japan, harmattans along African coasts, pamperos near Argentina, hurricanes in the South Seas—and had come through unscathed. She had visited the most unfrequented islands, and was without doubt better known the world over than any ship that sailed the seas." (p. 10).

In May 1928, she left the United States for her seventh and last voyage, a three year cruise of all oceans, under command of Captain JAMES PERCY AULT (1881-1929). This cruise was different from all preceding ones in that its program included oceanographic investigations of various kinds in addition to the magnetic and electric survey. It was very successful until Nov. 29, 1929 when a fatal accident in the harbor of Apia destroyed the good ship and caused the Captain's death.

The present volume is a very simple but moving account of that final

cruise, admirably illustrated with an abundance of photographs (almost 200). The daily routine of navigation and scientific research is described, the little incidents which brought some variety into it are briefly narrated, and we are given a few tantalizing glimpses into the crew's activities when a landing was made. For example, they visited Iceland, the mysterious Easter Island, the tomb of Mr. and Mrs. ROBERT LOUIS STEVENSON on Mount Vaea, near Apia, etc.

Unfortunately the historical part is very deficient, and we are shown (on p. 6) a portrait of HALLEY which I was not able to identify. When will the average scientist learn that a historical fact like any other is interesting only to the extent that it is accurate? Obviously the scientific staff was not very strong on history—even the history of its own subject. We are given an amusing proof of that on p. 94. The author, in describing the reception of the *Carnegie* in Hamburg, says:

> "Later on that morning we paid our first visit to the Deutsche Seewarte, a combined hydrographic and meteorological institution, proudly situated on a hill overlooking Hamburg harbor. This organization is justly famous for its progressive and meticulous research into problems of marine and aerial navigation. With one accord the members of the Seewarte staff shook our hands in the heartiest manner, as 'fellow countrymen of MATTHEW FONTAINE MAURY'! It must be admitted that some of us were mystified at first. As we came in we had noticed a bust prominently displayed at the entrance to the building, and this was pointed out to us as MAURY's statue. It only goes to show how a prophet is without honor in his own country. Had not MAURY created with his own hands the modern science of marine meteorology? Had he not made the first modern bathymetric charts? Had he not developed the electrically controlled submarine-mine as a defensive weapon in warfare?"

Just imagine this, a party of American scientists engaged in a three year's cruise round the world for oceanographic research—who did not know of their American predecessor, one of the founders of oceanography, and had to be told by German colleagues who he was! Innocents abroad. This should not be and only shows the necessity of some teaching on the history of science,—especially for scientists. An oceanographer should know MAURY (1), even as an American should know WASHINGTON. Of course we would hardly expect him, if not a historian, to know very much, but there is a minimum of knowledge, below which ignorance ceases to be funny and becomes sinful.

I would have liked also (but I admit I am a little hard to please) a more generous development of the theoretical part. Granted that the book was meant for the "general reader," it might have been made far more

(1) About him see CHARLES LEE LEWIS: Maury. The pathfinder of the seas (Annapolis, 1927; *Isis*, 11, 470).

instructive than it is. The intelligent reader should have been given a chance to learn a little more about those fascinating subjects—terrestrial magnetism and electricity—than the author was willing to impart. I cannot help feeling that he has not made the most of his great opportunity.

Even as it is, this book is a splendid monument to the members of the scientific and sailing staff of the *Carnegie*, and especially to Captain AULT, whose honor and privilege it was to die in the fulfillment of peaceful duty.

GEORGE SARTON.

Julius Ruska.—*Turba Philosophorum* : *Ein Beitrag zur Geschichte der Alchemie*. (Quellen und Studien zur Geschichte der Naturwissenschaften und der Medizin, herausgegeben vom Institut für Geschichte der Medizin und der Naturwissenschaften, redigiert von P. DIEPGEN und J. RUSKA, Band I.) 368 p. Berlin, JULIUS SPRINGER, 1931. (RM. 58).

As FERGUSON wittily expressed it, the *Turba philosophorum* is to be regarded as the supposed report of a meeting of a " Hermetic Association for the Advancement of Alchemy," with PYTHAGORAS as president and ARISLEUS as the convener or recorder. It shares with the *Tabula smaragdina*—of which also our detailed knowledge is mainly due to the brilliant work of Prof. RUSKA—the honour of being revered as a final authority by the alchemists themselves. Until the publication of the present book, however, very little accurate information on the nature and history of the *Turba* was available, and the most diverse views were expressed on the subject by SCHMIEDER, KOPP, BERTHELOT, WAITE, and other scholars. SCHMIEDER was inclined to date it in the twelfth century, but KOPP was not satisfied with the arguments adduced ; while WAITE, who described the *Turba* as " indisputably the most ancient extant treatise on Alchemy in the Latin tongue ", suggested that the Latin version might be a translation from the Arabic or Syriac. Mr. WAITE's English translation was published in 1896, but was not a critical edition, and can only be regarded as a praiseworthy pioneer effort to fill a serious lacuna in English alchemical literature.

Prof. RUSKA's edition is of a very different calibre. With his accustomed energy, backed by an unsurpassed knowledge of Arabic and Persian alchemical works, he has tackled the problem deeply, systematically, and from every point of view. As a result, the whole question has received quite unforeseen illumination, and in the reviewer's opinion there is very little of fundamental importance that remains to be said on the matter.

After a concise account of earlier work on the history of the *Turba* —from which, as any student of the subject must have found, very

few substantiated facts can be extracted—Prof. RUSKA deals fully with the suggestion that the treatise is ultimately of Arabic origin. He at once makes the important point that, to identify the numerous proper names in the text—names that appear to be Greek, yet Greek of no recognizable variety—we must assume them to have been Latin attempts to transcribe Arabic transcriptions of Greek names. The suggestion is not altogether new : from time to time such enigmas as *ethees* have been solved by adopting it. But Prof. RUSKA has shown that by applying it consistently to the names in the *Turba* the solution of so many mysteries becomes immediately apparent that its validity is unquestionable. That is not to say that considerable skill has not been required in applying the hypothesis ; indeed, it is here that Prof. RUSKA's versatile, but careful, scholarship appears to the greatest advantage and scores one of its most effective triumphs. Take, for example, such riddles as *Iximidrus*, *Pandolfus*, *Ascubofen* and *Horfolcus* : names that appear to have no connection with any known alchemists or philosophers, but to be the exotic fruit of a disordered imagination. Prof. RUSKA shows us that, on the contrary, they may all be identified with certainty. *Iximidrus* is merely a transcription of the Arabic ĪKSiMĪDRuS, which in turn is a faulty transcription (through wrong pointing) of the Greek ANAXIMANDROS. *Pandolfus* is the Latin scribe's version of BaNDaFiLuS, an easily-made error for ANBaDuQLiS or EMPEDOKLES ! Similarly, the queer name *Ascubofen* is shown to be a corrupted transcription of XENOPHANES, and *Horfolcus* of HERAKLEIOS. These identifications, thus briefly set down, may appear to be highly controversial ; but readers of Prof. RUSKA's book who are sufficiently familiar with the fortunes of Arabic proper names in medieval Latin transcriptions will be fully convinced by the cogency of his arguments, and will congratulate him upon his ingenious perspicacity.

Upon the same principle, many of the conundrums set by the names of substances given in the *Turba* may be solved with equal success. Thus *atitos* proves to be the *ethees* mentioned above, i.e. *aṭasiyūs* (Arabic) from the Greek λίθος ἐτήσιος; *corsufle* is none other than *kharsufla* in Arabic, i.e. the Greek " chrysocolla " ; *efflucidinus* has arrived in a mutilated form from the Greek κλαυδιανός and *gadenbe* is a similarly corrupt form of καδμία.

Further evidence that the Latin version of the *Turba* is based upon Arabic sources is to be found in the numerous Muslim formulae that occur in the book. The most striking instance is a passage in the " fifth speech of Arisleus " : *Dico tamen, quod Deus unus est, nunquam genuit nec genitus est*, which is obviously taken from the 112th. Sura of the Qur'ān, " Say : God is One ! God the Eternal ! He begets not and is not

begotten, nor is there any like unto Him!" It is rather amusing to find that in a later recension of the *Turba* the pious Christian scribe has altered the offending passage to *unumque genuit neque tamen genitus est*. Unmistakably Muslim, again, is the history of Creation given in the "ninth speech of Eximenus": *Deus suo verbo omnia creavit, qui dixit estote, et facta sunt*—; while turns of phrase and characteristic locutions here and there add cumulative support to a thesis that is, in fact, unimpugnable.

Admitted, however, that the Latin *Turba* is directly derived from the Arabic, the question nevertheless remains: Was the Arabic version an original work, or was it itself a translation from the Greek? Prof. RUSKA rejects the latter possibility, and says that, in addition to the kind of evidence already described, "können für die Originalität der arabischen Turba schon jetzt weitere und, wie ich glaube, durchschlagende Gründe geltend gemacht werden." These further grounds are, indeed, conclusive: the author of the *Turba* uses Arabic nicknames, quotes passages from other Muslim alchemical treatises, and evinces throughout an acquaintance with the early form of Arabic alchemy exemplified by the books of "Krates" and "Al-Ḥabīb." Prof. RUSKA would, in short, place the *Turba* later than these two books, but in the same tradition and series; he regards the 9th or 10th century as the earliest possible date for it, and Egypt as the most probable country of origin.

He next proceeds to review the printed editions of the *Turba* and to describe the three different recensions. A full account of the various manuscripts follows, and is completed by a description of various translations. Having prepared the ground so thoroughly, Prof. RUSKA next gives us a critical Latin text, based mainly upon the thirteenth century manuscript Qu 584 of the State Library at Berlin. This manuscript is exceptionally valuable in that it was very carefully corrected by the scribe himself; and in those places where uncertainty exists it can be collated with a printed version of the best recension. In the establishment of such texts as that of the *Turba*, Prof. RUSKA's experience has been very wide, and we may therefore confidently accept his conclusions as definitive. The consideration of textual *minutiae* lies outside the province of a reviewer, but it may at least be remarked that the present writer has found no instance where he would differ from Prof. RUSKA's judgement as to the correct reading.

To the general student of the history of science, the third part of the book will be the most attractive. It consists of a careful, well annotated German translation of the Latin text established in Part II. How difficult it is to give an intelligible version of early alchemical treatises, only those who have attempted the task can properly appreciate; and they will be

the first to express warm admiration at the masterly way in which Prof. RUSKA has accomplished a veritable *tour de force*. In his valiant study of Arabic alchemy, BERTHELOT more or less succeeded in producing comprehensible versions of the treatises with which he dealt, but only at the serious cost of arbitrarily adapting HOUDAS' translations. Prof. RUSKA has avoided this illusory satisfaction, and where his translation is difficult to follow, the difficulty arises solely from the obscurity of the original. That is the only sort of translation permissible if errors are not to be perpetuated, and scholars must thank him alike for the solution of many hard passages, and for his courage in recognizing that others are at present insoluble.

The fourth and concluding part of the book deals with the literary problem presented by the *Turba*. Here the interesting suggestion is made that "die Turba eine Kampfschrift gegen diese griechischen Alchemisten sein soll, die den Zweck hat, eine Befreiung der Kunst von der Pest der Decknamen herbeizuführen und sie auf den Boden einer allgemein anerkannten Naturphilosophie zu stellen." How far the attempt was successful students of alchemy will realize; but it is at least certain that the Arabs were the first to deal radically with the nuisance of "Decknamen." The "Decknamen" used by JĀBIR and RĀZĪ fall into a different category; for such names as "the flying slave" for mercury and "the eagle" for sal ammoniac were merely technical terms as intelligible to the average alchemist as sodium chloride and nitrobenzene are to the modern chemist. We cannot here follow Prof. RUSKA's arguments on this point, but they are as weighty and as lucid as usual. We must content ourselves by saying that, recalling the many books with which he has laid historians of science very deeply in his debt, the present volume must be universally welcomed as the *magnum opus* of a brilliant series.

E. J. HOLMYARD.

K. Ogura.—*History of mathematical education.* Iwanami Shoten, Tokyo, 1932. IV+356 pp. Price 2.50 yen. (in Japanese).

As indicated by the title, Dr. OGURA's book opens a new field—the history of mathematical education. The author's purpose includes more than a mere re-editing of available data. He apparently wishes to correlate from an objective standpoint the data on mathematical education not only with mathematical progress, but also with social, economic, and political history, insofar as they are pertinent to the general development of civilization.

In an objective analysis of social life it is necessary first to establish a background based on the psychology of education, to decide the nature

of the problem and its context, and then proceed to a method of teaching. Scientific research in mathematical education in this strict meaning was not begun, however, until the twentieth century. This problem will be treated in a sequel to the book reviewed here, *History of modern education in mathematics*, which will be published shortly. In a sense the volume under review is a preparatory introduction to a history of mathematical education, the main emphasis therein being placed on mathmeatical education in secondary schools.

This review will be concerned chiefly with the last two chapters, but the scope of the other four is indicated by the chapter headings as listed : I. Period of Cultural Studies (centering around the sixteenth century) ; II. Development of Realistic Education (about 1600-1750) ; III. Transition Period of Education ; Establishment of Mathematical Courses (about 1750-1840) ; and IV. Period of Decline in Mathematical Education (about 1840-1900).

Chapter V is devoted to Japanese education during the last stage (1800-1872) of the Feudal Age. During the feudal era under the Tokugawa shoguns (1603-1868) the science of mathematics, which had been permitted to die after its introduction from China through Korea in the sixth century, was restored to favor for a second time. Mathematics flourished under the able hands of such men as SEKI (1642 ?-1708), TAKEBE, and YASUJIMA. Unfortunately, however, the knowledge was not propagated, because there was a tendency on the part of mathematicians to keep their methods secret. Most of them tried to build their own traditions, opening private schools and fostering protégés.

Japanese mathematics was developed without the aid of the Feudal government, because sharp lines of social distinction among the nobility and the common people resulted in a different type of education for each group, practical mathematics being relegated to the latter. Outside influences, however, resulting from the Western economic movement, began to invade Japan and, closely following the Opium War (1840), the Japanese were introduced to foreign science through Chinese translations of the lectures of WYLIE ALEXANDER and of books by DE MORGAN and E. LOOMIS. The assimilation and development of mathematical science were accelerated by their coincidence with pressing considerations of national defense.

Chapter VI gives an analysis of the constructive period (1872-1902) of education in mathematics in Japan. The first step was a compulsory unification of the educational system made possible by that most important political and social revolution which resulted in the Meiji restoration. In 1871 a Department of Education was created for the advancement of learning. The first Code of Education, based on the highly standar-

dized French school system, was promulgated in 1872. Immediately thereafter an American, SCOTT, was invited to the Normal School, forerunner of the Tokyo Bunrika Daigaku, to introduce the American methods of teaching English and arithmetic. Following SCOTT came DAVID MURRAY (1830-1905), who, during his stay of five years, placed American methods of teaching mathematics in a preeminent position in the educational program. Students, however, were severely handicapped because teachers for the most part were foreigners and text-books were translated literally. It became necessary to revise the Code, to reduce the number of hours allotted to arithmetic and geometry, and to add old Japanese mathematics to the curriculum.

When KIKUCHI returned from England, he effected the substitution of the English school (and of TODHUNTER in particular) for the American influence which had reached a climax around 1877 through the books of H. N. ROBINSON and of CHARLES DAVIES. Not many years afterward the industrial revolution which followed the Sino-Japanese War created a capitalistic class and made possible a sudden expansion of secondary education. When KIKUCHI gave up teaching to become Minister of Education, he set the stage for FUJISAWA to carry out unhampered his program of reconstruction. It was unfortunate, however, that KIKUCHI and FUJISAWA were not closely in touch with the change in educational methods brought about by PERRY in England and by MOORE in America. Minister KIKUCHI and Professor FUJISAWA were sound in their methods and sincere in their conviction, but they were adhering to the very traditional English method which PERRY at that time was attempting to break down. Dr. OGURA brings his discussion to a close with a review of this trend.

A layman with a little knowledge of mathematics will find the book interesting and instructive. This book should be especially helpful for middle school teachers of Japan (who are kept busy preparing their students for the competitive examinations for higher institutions) in instilling an appreciation of the background along with the routine training process. This review of Dr. OGURA's book is necessarily brief, and doubtless inadequate, but if it direct the attention of the historian of science to the field of the development of mathematical education as presented in the book, it will have served its purpose.

(*Cambridge, Mass.*) S. IKEHARA.

George Herbert Mead (1863-1931).—*The Philosophy of the Present.* xi+199 p. Open Court, Chicago, 1932. $ 3.00.

The present volume, the third to be published under the Paul Carus Foundation, is the posthumous work of GEORGE HERBERT MEAD, professor

of philosophy at the University of Chicago, whose death occurred on April 26, 1931. The book contains prefatory remarks by JOHN DEWEY and an extensive introduction by ARTHUR E. MURPHY who undertook the task of preparing for publication the material in the form left by the author at is death.

The book consists of four chapters and five supplementary essays : The Present as the Locus of Reality, Emergence and Identity, The Social Nature of the Present, The Implications of the Self, Empirical Realism, The Physical Thing, Scientific Objects and Experience, The Objective Reality of Perspectives, The Genesis of the Self and Social Control.

The title would suggest that the book is devoted to current philosophical ideas, but the word *present* refers rather to the problem of the reality of the present moment in its relationship to the emergent future and the vanishing past. The thought of the author is strongly dominated by the philosophy of relativity which he elucidates in a competent manner with sufficient detail for his purpose. One should make special note that Professor MEAD particularly stresses a fact too feequently passed over by those who would trace the philosophical implications of relativity, namely, that the theory of *special* relativity was written fundamentally in the equations of MAXWELL as a property of their invariance long before EINSTEIN was led to his discoveries through attempts to interpret the MICHELSON-MORLEY experiment.

The main theme of the essay is that only the present has *reality* and that the future is a new and emergent thing which flows from the present and the past, but is not completely determined by them.

Thus in the opening paragraph the author says :

> " The subject of this lecture is found in the proposition that reality exists in a present. The present of course implies a past and a future, and to these both we deny existence."

This denial of reality to the future is essentially a denial of determinism, particularly as this philosophical doctrine applies to society. Thus the author continues at another place in the book :

> " Now the principle of sociality that I am attempting to enunciate is that in the present within which emergent change takes place the emergent object belongs to different systems in its passage from the old to the new because of its systematic relationship with other structures, and possesses the characters it has because of its membership in these different systems."

The ancient problem of determinism thus runs as a red thread through the entire thought of the author. Fortified by the discoveries in molecular

physics and by the HEISENBERG principle of indeterminacy, Professor MEAD believes that no knowledge of the past is sufficient completely to define the present or to predict the future.

> " In the analysis which I have undertaken," he says, " we come then, *first*, to passage within which what is taking place conditions that which is arising. Everything that is taking place takes place under necessary conditions. *Second*, these conditions while necessary do not determine in its full reality that which emerges. We are getting interesting reflections of this situation from the scientist's criticism of his own methods of reaching exact determination of position and velocity and from the implications of quanta. What appears in this criticism is that while the scientist never abandons the conditioning of that which takes place by that which has gone on, expressed in probability, he finds himself quite able to think as emergent even those events which are subject to the most exact determination. I am not attempting to previse what later interpretation will be put upon the speculations of DE BROGLIE, SCHROEDINGER, and PLANCK. I am simply indicating that even within the field of mathematical physics rigorous thinking does not necessarily imply that conditioning of the present by the past carries with it the complete determination of the present by the past."

This profound problem has its basis, perhaps, in the foundations of the theory of probability, particularly as this theory applies to the kinetic theory of small particles. In molar physics there is no question about the power of prediction to within observational limits. Jupiter will follows its charter path in the sky ; the anomalies of the motion of the moon await only the introduction of more terms in the perturbing function in order to be completely explained away. But in molecular physics the story is otherwise. Statistical swarming, the law of large numbers, and other essentials of determinism give way to individual impacts of photons and electrons. Complete indeterminacy holds sway. It seems to the reviewer that the question is not primarily one as to whether the universe is deterministic or otherwise, but rather as to the existence of a critical point somewhere between the microcosm and the macrocosm where the determinism of large numbers must yield to the indeterminacy of individuals.

It seems rather unfortunate to the reviewer that those who write upon the problem of the future in its relationship to the past so frequently omit any reference to the fruitful concept of " hereditary mechanics " which appears throughout the writings of VITO VOLTERRA, particularly in his notable treatise : " Leçons sur les fonctions de lignes," published in Paris in 1913. The enunciation of this principle and its contrast with the non-hereditary principle that the future of a dynamical system is completely determined by the boundary conditions of the present is found in an article by E. PICARD : " La mécanique classique et ses approximations successives," *Scientia*, vol. 1 (1907), pp. 4-15. The reader is

also referred to the illuminating comment by M. WINTER in an article : " Time and Hereditary Mechanics " published in the *Monist*, vol. 35 (1925), pp. 70-80. The connection between this important principle and the ideas of Professor MEAD is easily seen in the following quotation from PICARD :

" In all this study (of classical mechanics) the laws which express our ideas of motion have been condensed into differential equations, that is to say, relations between variables and their derivatives. We must not forget that we have, in fact, formulated a principle of *non-heredity*, when we suppose that the future of a system depends at a given moment only on its actual state, or in a more general manner, if we regard the forces as depending also on velocities, that the future depends on the actual state and the infinitely neighboring state which precedes. This is a restrictive hypothesis and one which, in appearance at least, is contradicted by facts. Examples are numerous where the future of a system seems to depend on former states : here we have *heredity*. In some complex cases one sees that it is necessary, perhaps, to abandon differential equations and consider functional equations in which there appear integrals taken from a distant time to the present, integrals which will be, in fact, this hereditary part. The proponents of classical mechanics, however, are able to pretend that heredity is only apparent and that it amounts merely to this, that we have fixed our attention upon too small a number of variables. But the situation in this case is just as it was in the simpler one, only under conditions that are more complex."

It will thus be seen that the problem of the present in its connection with the emergent future is one upon which physicists under the strict formulizations of their mechanics cannot agree. How much more difficult is the general problem in which the variables embrace the field of human society.

It will appear from this that Professor MEAD has written a stimulating volume which will take its place in the growing literature of determinism.

(*Indiana University.*) H. T. DAVIS.

Robert T. Gunther.—*The astrolabes of the world.* Based upon the series of instruments in the Lewis Evans Collection in the Old Ashmolean Museum at Oxford, with notes on astrolabes in the collections of the British Museum, Science Museum, Sir. J. FINDLAY, Mr. S. V. HOFFMAN, the Mensing Collection, and in other public and private collections. Volume I. The Eastern Astrolabes. XVII+III+304 p., 139 figs., 68 pl., frontisp. Oxford, University Press, 1932. (£ 10.10.0 for the two vols.).

The death of CARL SCHOY (1877-1925) and of JOSEPH DRECKER (1853-1931) interrupted the realization of an important project : the writing of a history of the astrolabe. Both as a scientific instrument and as an *objet d'art*, the plane astrolabe deserves at least the same attention as,

e.g., celestial and terrestrial globes, sundials, clocks, etc. The publication of a corpus of all the important treatises on the astrolabe is, of course, a necessary preliminary step in the writing of a history of the instrument which played such an essential part in the preservation, during the Middle Ages, of Greek astronomical knowledge. On the other hand, the usefulness of a descriptive illustrated catalogue of all the accessible surviving specimens is quite obvious; no detailed classification of the astrolabes can claim to be complete unless it takes into consideration even the hybrids and freaks produced by the vicissitudes of the individual instruments or by the idiosyncrasies of the craftsmen. However, both the corpus and the catalogue are but source material; their publication does not constitute a history of the astrolabe; their analysis and integratio by a scientist and a scholar—preferably of the BASHFORD DEAN type—must follow; it took a palaeoichthyologist to study the "natural history" of armor, and it will take more than a mere astronomer or orientalist to study the evolution of the scientific works of art called astrolabes; WILLIAM HOOK MORLEY (1815-1860) was such a man, but he had access to a few dozens of astrolabes, not to several hundreds.

The book under review consists of an excellently illustrated *catalogue raisonné* of a considerable number of astrolabes, and of a small group of mediaeval and modern writings on the subject inserted as introductions to some of the chapters; it is more than a mere catalogue of the LEWIS EVANS collection of which the author is the curator; it is less than an exhaustive or even representative compilation of the source material indispensable for the writing of a long overdue historical treatise on the astrolabe. The author calls it, in the preface, "a general catalogue of the astrolabes of the world."

If one agrees, in principle, with the use of reprints and translations of important treatises as prefatory material to the various chapters,—an easy way of compiling impressive volumes,—there remains the question of selection. The volume under review offers, as an introduction to the chapter on Greek astrolabes, the first English translation, by competent scholars, of the Greek treatise by JOHN PHILOPONOS, and of the Syriac treatise by SEVERUS SĒBŌKHT (middle of the VIIth century).

GUNTHER's assignment (p. 59, 61) of JOHN PHILOPONOS of Alexandria and of his treatise on the astrolabe to the first half of the VIIth century is wrong; JOHN PHILOPONOS wrote his treatise in the first half of the VIth century. MAX MEYERHOF disposed of the VIIth-century JOHN in his paper on JOHANNES GRAMMATIKOS (PHILOPONOS) von Alexandrien und die arabische Medizin (*Mitteilungen d. Deutschen Instituts f. aegypt. Altertumskunde in Kairo*, 2, 1-21, 1931; see also *Isis*, XVIII, 447-48, 1933). But even while the ghost of the VIIth-century JOHN was

worrying the historians of medicine, the treatise on the astrolabe was correctly assigned to the first half of the VIth century, the lifetime of the only real JOHN PHILOPONOS of Alexandria. GUNTHER—and his translator of PHILOPONOS' treatise—should have consulted not only the antiquated HASE edition of 1839, but also the studies and translations by TANNERY and by DRECKER (the first German translation, with introduction and notes, appeared in *Isis*, XI, 15-44, 1928).

It seems to the reviewer that the first English translation of another Greek treatise already accessible in DRECKER's German translation (*Isis*, IX, 255-278, 1927) should have been included; although TANNERY was undoubtedly right in calling the Planisphaerium a "très médiocre ouvrage" (Recherches sur l'histoire de l'astronomie ancienne, p. 52, 1893), the treatise remains the earliest extant exposition of the principles of stereographic projection, and deserves to be included in a representative collection of early writings on the theory, construction, and use of the astrolabe. Incidentally, it ought to be investigated whether this Greek-Arabic-Latin Planisphaerium is identical with PTOLEMY's lost treatise on the "spreading out of the surface of the sphere"—*ἁπλωσις ἐπιφανείας σφαίρας*—mentioned by SUIDAS; PHILOPONOS seems to refer to a Ptolemaic treatise on the astrolabe which is not identical with the Planisphaerium.

Another disappointment meets the reader who looks for a prefatory treatise in the chapter on "Arabian Astrolabes": it is absent. The first English translation of the IXth-century treatise by 'ALI IBN Ī'SĀ (German translation by SCHOY, *Isis*, IX, 239-254, 1927), could be suggested as an appropriate introduction. Important passages from the Latin translation, by JACOB IBN MAḤIR (PROPHATIUS) and JOHN OF BRESCIA (1263), of AL-ZARQĀLĪ's treatise on the ṣafīḥa are quoted in the chapter entitled "Moorish Astrolabes."

The most valuable, among the prefatory reprints included in the volume under review, is a reduced facsimile reproduction of WILLIAM H. MORLEY's Description of a planispheric astrolabe (London, 1856). MORLEY's work—50 p., 21 pl., f°—certainly deserved a wider distribution than the 100 copies of the original edition permitted; after the lapse of three quarters of a century, it remains the best modern treatise on the astrolabe; it contains not only a detailed description of the astrolabe constructed (A. H. 1124 = A.D. 1712) for shah HUSAIN, but also an excellent "account of the astrolabe generally, with notes illustrative and explanatory: to which are added, concise notices of twelve other astrolabes, Eastern and European, hitherto undescribed." MORLEY's scholarly work offers a wealth of Persian, Arabian, and Indian source material, combined with a scientific exposition of the geometrical theory and the mechanical features of a small but representative group of instruments. The plates

added to MORLEY's model monograph prove that the scholar and scientist was also a skillful and resourceful craftsman; four of them are good examples of his draftsmanship; the origin of the seventeen plates illustrating the HUSAIN astrolabe is best described in MORLEY's own words: "When I first saw the beautiful Astrolabe hereinafter described, it struck me that, as its engraved surfaces were all plane, its several parts might be readily and exactly reproduced by the Anastatic method of printing... The experiment was made, and the Plates at the end of this volume are the result."

The success of a comparative study of a large number of specimens depends on a well devised system of classification. Considered as scientific instruments, astrolabes could be classified by their astronomical or mechanical features; this subdivision in genera and species would necessarily cut across some of the geographical or cultural borders, but it would afford a chronological exposition of the evolution of the instrument; a classification of this kind would probably satisfy the historian of science. On the other hand, it is possible to group astrolabes by their stylistic or even by their epigraphic features; this method of subdivision, if and when practicable, neglects generic and specific characteristics of the instruments, and leads to the consideration of chronological sequences within the geographical or cultural groups; a classification of this kind might possibly appeal to the historian of art.

The chapter titles in the volume under review—Greek, Byzantine, Oriental, Persian, Indian, Hindu, Arabian, Moorish, and Jewish astrolabes,—and the chapter titles in volume II—Spanish, Italian, French, Flemish and Dutch, German, and English astrolabes,—suggest a geographic-cultural classification of the specimens. The somewhat confusing chapter title, "Oriental Astrolabes," occupies, in the table of contents, an equal rank with its neighbors, the "Byzantine Astrolabes," and the "Persian Astrolabes"; in the book itself, it has a cryptic subtitle, "The Arabians," and heads a general introduction, consisting of four pages, to the following chapters of the volume. From this chapter on "Oriental Astrolabes" we quote the following lines throwing some light on the principles of classification adopted (p. 112):

"For convenience we propose to describe our Oriental Astrolabes in chronological order in four provisional main groups, the typical members of each of which are not difficult to distinguish, and are associated with the four geographical regions.

1. Persian.
2. Indian.
3. Arabian, Syrian, Mesopotamian, Egyptian.
4. Moorish, including N.W. Africa and Hispano-mauresque.

Although it is usual to describe the astrolabes of the third group as 'Arabian,'

the name must be taken to refer to the language in which they were inscribed, rather than to imply that there were factories of instruments among the burning sands of Arabia in the strict geographical sense. Astrolabes that were actually made in Mecca or any of the towns of Yemen or the Hadramaut are rare. The skilled workers in metal lived in Damascus, Bagdad, Cairo, and other towns in countries that were conquered by the Arabs, and wisely governed by them until the fall of the Califate of Bagdad in A.D. 1258. During the previous five centuries was the floruit of the 'Arabian' Astrolabe."

This classification of "Oriental Astrolabes" according to their "association with the four geographical regions" is not as simple as it may seem. Take, e.g., "Professor WILSON's Astrolabe" (p. 186-87) described in the chapter on "Indian Astrolabes": "The oldest known astrolabe made, or perhaps we should say, modified for use in India, belongs to the period of the Mohammedan dominion, and was essentially an instrument of the Arabian type inscribed in Cufic which has been altered and reinscribed for use in India... The list of Indian cities on the interior of the *umm* is more modern, being engraved in Devanagari characters." The chapter on "Hindu Astrolabes"—as distinguished from "Indian Astrolabes"—is hardly consistent with the association of oriental instruments with "four geographical regions"; the use of Devanagari script is hardly a sufficient reason for their isolation. The difficulties of the adopted classification are acknowledged by the author himself in the one-page introduction to the chapter on "Arabian Astrolabes" (p. 229):

"The grouping of astrolabes produced in such widely separated geographical regions as Arabia, Syria, Mesopotamia, and Egypt under a single head has already been suggested on page 112, and in the present imperfect state of our knowledge, we find some justification in the fact that even experts who had been entrusted with the arrangement of the exhibits at the recent International Persian Exhibition in London had difficulties in distinguishing between the metal-work of Persia and of Mosul, and this is hardly remarkable when it is remembered that the craftsmen themselves migrated from town to town."

If the reader hopes to find a clear statement of the stylistic or epigraphic differences between the "Moorish" and the "Arabian" astrolabes of the quadripartite classification, he will be disappointed. Here is a quotation from the introduction to the chapter on "Moorish Astrolabes" (p. 250):

"Moorish astrolabes, and especially those made in Spain at that early period, are characterized by certain readily recognizable features that distinguish them from Persian instruments of the same epoch. Their ankabuts are designed on a simpler plan. There are none of those elaborate intertwinings and serpentine tracery that give the astrolabes of the farther East their attractiveness and grace.

But the Moorish instruments have a charm of their own, which is derived more particularly from the curvature and decoration of the star-pointers, than from any artistic treatment of supporting bands and bars, which in general are studiously plain. When ornament is introduced, it is frequently of a type suggested by the Moorish arch with its ogee point and cusped sides. In the lettering the Cufic prevails. The *kursi* or bracket is frequently far smaller than is usual in Eastern astrolabes. There is no throne inscription. *Mudirs* may be multiplied. Star-pointers are frequently embellished with bright silver knobs and perforate bases."

The chapter on " Moorish Astrolabes " includes the description of the instrument " stated to have once belonged to REGIOMONTANUS " (p. 280), with Latin inscriptions ; on the other hand, two astrolabes with " inscriptions in Hebrew characters " are described in a separate chapter of two pages, " Jewish Astrolabes," at the end of the volume—a chapter which apparently is independent from the quadripartite classification. To add to the confusion created by the adopted system, the author has inserted what looks like an independent chapter, " The Arzachel," into the chapter on " Moorish Astrolabes."

The illustrations of this general catalogue of astrolabes are excellent, and will be of great assistance both to historians of science and to historians of art. The number of specimens dealt with in this volume is about 160; some of these astrolabes are of exceptional beauty or of considerable scientific interest.

The only item described in the chapter on " Greek Astrolabes " is " The astrolabe of Anticythera," ca. A.D. 250. This intricate piece of mechanism, one of the neglected gifts of Poseidon to Athens in the winter of 1900/01, was undoubtedly an astrononomical instrument and time-measuring device ; it was described by PERICLES REDIADES in J. N. SVORONOS' Das Athener Nationalmuseum, I, 43-51, 1908 ; it is dated by a partly legible inscription giving instructions for its use. GUNTHER failed to mention that SVORONOS has established (*loc. cit.*, p. 49/50) striking analogies between the wording of the inscription on the instrument from Anticythera and the text of the treatise of JOHN PHILOPONOS. GUNTHER calls attention to a XIIIth-century Persian astrolabe (described on p. 118) in the LEWIS EVANS collection which has " a geared calendar movement of cog-wheels within it " ; a more detailed study of this mediaeval mechanism might throw new light on the instrument from Anticythera.

The short chapter entitled " Byzantine Astrolabes " contains a description of the only specimen known—the XIth-century instrument in the Brescia museum. " Both from the point of view of its art, and from that of its lettering, it shows no trace whatever of having been in any way modified by Arab or Persian influence... We may perhaps have

here the sole survival of a scientific instrument embodying Greek ideas..." These statements are but partly true. GUNTHER should have called the attention of his readers to the fact that the plain *arachne*, and especially the dagger-shaped star pointers of the Brescia astrolabe prove its close kinship to instruments of the same period classified as "Persian" and "Arabian," such as the oldest known dated astrolabe (Persian; A.H. 374 = A.D. 984; LEWIS EVANS collection) or the astrolabe of Pope SYLVESTER II (Arabian; ca. A.D. 990; Tribuna di GALILEO, Florence). Whether these early instruments were direct descendants of lost Byzantine astrolabes, or whether the Brescia astrolabe is a Persian article manufactured for export to Byzance, the fact remains that not a single Greek or unquestionably Byzantine astrolabe has come down to us.

To conclude : a hastily compiled and rather carelessly proofread catalogue de luxe. Caveat lector.

ALEXANDER POGO.

Willem de Sitter.—*Kosmos*. A Course of Six Lectures on the Development of our Insight into the Structure of the Universe, delivered for the Lowell Institute in Boston, in November 1931. Cambridge, Massachusetts, Harvard University Press, 1932. VIII+138 pages. 7 portraits + 5 full-page plates. Price, $ 1.75.

It is only within the last dozen years that we have fully realized the immensity of the universe. Even a man in the street can hear astronomers talking about nebulae which are a hundred million light years away, but very few pause to consider that this knowledge is of a very recent origin. As recently as 1919 Sir JAMES JEANS, in his book "Problems of Cosmogony," estimated the distance of the Andromeda nebula at 5400 light years, but now we think that it is 870,000 light years away from us. It has truly been said that we may or may not agree with the relativists who believe in the expanding universe, but nobody will question that our knowledge is rapidly expanding.

Dr. DE SITTER, Director of the Observatory and Professor of Astronomy at the University of Leiden, gives us a picture of this rapidly expanding knowledge of the universe. His book consists of six chapters the contents of which can be briefly summarized as follows : (1) Introduction ; The Ancients, PTOLEMY, COPERNICUS ; (2) KEPLER, GALILEO, NEWTON ; (3) WILLIAM HERSCHEL ; (4) KAPTEYN ; (5) Galactic and Extra-galactic systems ; (6) Relativity.

The most important omission in the historical sections of the volume concerns the Babylonian astronomy. Apparently the writer is unaware of the work done by KUGLER, EPPING, FOTHERINGHAM and others in this

field. In fact two of the greatest astronomers of all ages, the Babylonians NABURIANNU and KIDINNU, are not even mentioned. This is especially striking as the author goes rather into detail so far as Greek astronomy is concerned and mentions half a dozen minor Greek astronomers. The discovery of the precession of the equinoxes is attributed without reservations to HIPPARCHUS (p. 14) although there is much evidence now that it was known to the Chaldeans.

On page 18 the following curious statement can be found : " PTOLEMY was the last Greek astronomer. Nothing was added to science after his time, and Greek civilization was finally destroyed by the burning of the library of Alexandria by the Christian mob in A.D. 389." The reader may well puzzle over such a positive statement as to just where and when and why Greek civilization ended. For the author of this book the time between PTOLEMY and COPERNICUS is only " the dark ages of orthodox ignorance " (p. 18).

In the second chapter TYCHO BRAHE is called a Swedish nobleman (p. 19), although the excellent portrait on the next page has the inscription " Effigies Tychonis Brahe... Dani." Of course it is true that TYCHO BRAHE was born in what is now a part of Sweden, but at that time it was a part of Denmark, and had always been from time immemorial.

On page 28 the year of NEWTON's birth is erroneously given as 1643 instead of 1642. The author justly rejects the old fable that the publication of the " Principia " was delayed because NEWTON did not know the exact radius of the earth.

It may be advantageous for the purpose of a short series of public lectures to select a few great figures in a science like astronomy and concentrate attention on them. However, from the historical viewpoint this method is rather dangerous. The reader gets the idea from chapters 3 and 4 that after NEWTON there were only two astronomers of any consequence : Sir WILLIAM HERSCHEL and KAPTEYN. In fact the author says plainly : " About the year 1800 the astronomical world, at least so far as sidereal astronomy is considered, consisted of WILLIAM HERSCHEL alone " (p. 86). And on page 76 : " KAPTEYN's work has put its mark upon the whole of modern astronomy." To be sure, KAPTEYN's influence on the development of our ideas on the structure of the universe has been great, but this is not " the whole of modern astronomy." The composition of the sun and of the stars, properties of nebulae, comets, planets, meteors, evolution of the stars, all these subjects are as much parts of the Cosmos as the structural details of the Milky Way. In vain would one try to find a reference to most of these topics even in the next chapter devoted to photography and spectroscopy.

The closing chapter is on " Relativity " and the modern theories of

the universe. Being a relativist the author does not hesitate to say : " The conflict apparently is between the modern theories of stellar evolution... on the one hand, and the general theory of relativity on the other hand. If this were the real contest, there could be no doubt about the issue : the theory of relativity would come out victorious, and the theories of evolution would have to be revised." This means that the expansion of the universe may be so rapid as to leave no time for the development of stars, or even of our own earth. Unfortunately for this aspect of the theory of relativity we have a reliable estimate of the age of terrestrial rocks which proves to be something like two or three billion years. One is reminded of the advice to geologists to revise the age of the earth, which was given by Lord KELVIN when he found from his theory of contraction that the age of the sun was of the order of twenty million years.

Professor DE SITTER is too much of a scientist, however, to put absolute trust in theory. The reader is warned that " after all, the universe is an hypothesis... What we observe are the stars and nebulae constituting ' our neighborhood.' All that goes beyond that, in time or in space, or both, is pure extrapolation." (p. 133-134).

The book is certainly a good description of the current views on the structure of the universe with some references to older theories. It is not a history of the development of man's ideas about the universe, and probably was not meant to be such in spite of its subtitle. Such a task could not be done in six lectures. The reader will find this book interesting, although in places rather difficult.

Perkins Observatory N. T. BOBROVNIKOFF.
Delaware, Ohio.

Mark H. Wodlinger.—*Bibliographical Survey of Vitamins*, 1650-1930, with a Section on Patents. Compiled by ELLA M. SALMONSEN, Medical Reference Librarian, The John Crerar Library, Chicago, published by the author. 1932. (Price $ 10.00).

The author has assembled 10,098 titles dealing directly or indirectly with vitamins, and 240 citations of American and foreign patents involving discoveries relating to vitamins. The first reference is to GLISSON's (1650) paper on *The Rickets*. It was not until 1910 that the references became frequent enough to assemble them in annual groupings and not till 1915 that vitamins A-D qualified for separate classification. Vitamin E arrives in 1921. No citation of Capt. JAMES COOK's methods of preventing scurvy is given. In his *Journal* of the first voyage, under date of April 13, 1769, he writes as follows : " At this time we had but very few men upon the sick list, and these had but slite complaints.

The Ship's company had in general been very healthy, owing in a great measure to the Sour kroutt, Portable Soup and Malt ; the two first were served to the People, the one on Beef Days and the other on Banyan Days. Wort was made of the Malt, and at the discretion of the Surgeon given to every man that had the least simptoms of Scurvy upon him. By this means and the Care and Vigilance of Mr. MONKHOUSE, the Surgeon, this disease was prevented from getting a footing in the Ship. The Sour Kroutt, the Men at first would not eat it, until I put it in practice —a method I never once Knew to fail with seamen—and this was to have some of it dressed every day for the Cabin Table, and permitted all the Officers, without exception, to make use of it, and left it to the Option of the men either to take as much as they pleased or none at all ; but this practice was not continued above a Week before I found it necessary to put every one on board to an allowance ; for such are the Tempers and disposition of Seamen in general that whatever you give them out of the common way—altho' it be ever so much for their good—it will not go down, and you will hear nothing but murmurings against the Man that first invented it ; but the moment they see their superiors set a value upon it, it becomes the finest stuff in the world and the inventor an honest fellow. Wind easterly."

December 2, 1932. C. A. KOFOID.